University of Fort Hare

MAT 303

Supplementary Examinations: January 2019

Subject: Mathematics 3
Paper: Real Analysis

Time: 3 Hours | Marks: 100 | Subminimum: 40

This question paper consists of 5 pages

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Instructions

Attempt **NO more** than **FIVE**(5) questions. Symbols used have the usual meanings.

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Mathematics Supplementary Exams November 2018

Course Code: MAT 303 **Course Name:** Real Analysis
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Instructions: Answer **FIVE** questions. **This paper consists of 5 pages**

Question One

1.1 (a) Define a metric on a non-empty set X. (2)
(b) Define the diameter of a set A in $\mathbb{R}^n$. Hence find the diameter of the set $\{(x,y) : \frac{x^2}{25} + \frac{y^2}{32} \leq 1\}$ in $\mathbb{R}^2$. (3)
(c) Define the trivial metric on X. (1)
(d) Define the usual metric on $\mathbb{R}^n$. (1)

1.2 Let d be a metric on X, $A \subset X$ and $p \in X$. The distance between p and A is $d(p, A) = \inf\{d(p, a) : a \in A\}$.
Let d be the trivial metric on $\mathbb{R}$ and $A = (-2, 5) \subseteq \mathbb{R}$. (a) Find $d(5, A)$. If ρ is the usual metric on $\mathbb{R}$, (b) find (with motivation) (i) $\rho(5, A)$ and (ii) $\rho(-3, A)$. (3)

1.3 Show that, in $\mathbb{R}^2$, the set $A = \{(x,y) : 0 < y < 3\}$ is open. Draw also a rough sketch of set A. (4)

1.4 (a) Define the boundary of a subset A of $\mathbb{R}^n$. Then give another characterization of a boundary. (2)
(b) Let $A = \{(x,y) \in R^2 : -1 < x \leq 1\}$. Find the boundary of A. Draw set A. (3)
(c) Show that a boundary point need not be an accumulation point. (2)

1.5 (a) Show that the set $A = \{1 + \frac{1}{n} : n \in N\}$ is not closed in $\mathbb{R}$. (2)

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Question Two

2.1 If $x = \sup S$, for $S \subseteq \mathbb{R}$, show that $x \in \overline{S}$. (3)

2.2 Prove that a set $A \subset \mathbb{R}^n$ is closed if and only if for every sequence $\{x_k\}$ in A which converges, the limit lies in A. (4)

2.3 Let $B \subset \mathbb{R}^n$. Prove that $x \in \overline{B}$ if and only if there is a sequence $\{x_k\}$ in B converging to x, where $\overline{B}$ is the closure of B in $\mathbb{R}^n$. (5)

2.4 A metric space (M , d) is complete if every Cauchy sequence in M converges to a point in M. Show that the set $\boldsymbol{Q}^c$ of all irrational numbers (with the usual metric) is not a complete metric space. (2)

2.5 (a) Let $A \subset \mathbb{R}$, x, $y \in A$. Define what is meant by a path joining x to y. (1)

(b) Let $A \subset \mathbb{R}^n$. Define what is meant by (i) A is compact, (give three equivalent statements),
(ii) A is path-connected. (4)

2.6 Show that $A = \{x \in R^n : ||x|| \leq 4\}$ is path-connected. (3)

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Question Three

3.1 (a) The set $S = \{(x, y) \in \mathbb{R}^2 : -1 < x < 2\}$ is not compact. Why? (1)
(b) Let $A = \{1 + \frac{1}{n} : n = 1, 2, \cdots\}$. Is A compact? Explain. If A is not compact, how can we "compactify" it? (2)

3.2 The set $\{(1 + \frac{1}{n}, 3) : n = 1, 2, \cdots\}$ is an open cover of $(1, 2]$. Show that this cover has no finite subcover of A. (4)

3.3 Show that $[2, 3] \cap \boldsymbol{Q}$ is not path-connected, where $\boldsymbol{Q}$ is the set of all rational numbers. (3)

3.4 Prove that a set $A \subset \mathbb{R}$ is connected if and only if A is an interval. (4)

3.5 Let $f : A \to \mathbb{R}^n$. Prove that the following statements are equivalent:
(i) f is continuous on A;
(ii) For each convergent sequence $\{x_n\}$, converging to $x_0 \in A$, the sequence $\{f(x_n)\}$ converges to $f(x_0)$;
(iii) For each open set $U \in \mathbb{R}^n$, $f^{-1}(U) \subseteq A$ is open relative to A, i.e. $f^{-1}(U) = V \cap A$ for some open set V in A. (8)

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Question Four

4.1 Let $f : A \to \mathbb{R}^n$ be a continuous function. Prove that
(a) If $K \subseteq A$ and K is connected, then $f(K)$ is connected. (4)
(b) If $B \subseteq A$ and B is compact, then $f(B)$ is compact. (3)

4.2 Let $f(x) = 1/x$ where $f : (0, \infty) \to \mathbb{R}$. Show, from definition, that f is continuous at $x_0 \in (0, \infty)$. (3)

4.3 Let $f : [0, 1] \to [0, 1]$ be continuous. Show that f has a fixed point. (3)

4.4 (a) Find a continuous map $f : \mathbb{R} \to \mathbb{R}$ and a compact set $K \subseteq \mathbb{R}$ such that $f^{-1}(K)$ is not compact (2)
(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and let $A = \{f(x) : ||x|| = 3\}$. Show that A is a closed interval. (3)

4.5 Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous. Show that $\{x \in \mathbb{R}^n : ||f(x)|| < 2\}$ is open in $\mathbb{R}^n$. (3)

4.6 Let $\{f_k : A \to \mathbb{R}^m\}$ be a sequence of functions.
(a) Define what is meant by the sequence of functions converges **uniformly** to f. (2)
(b) Let $\{f_k : \mathbb{R} \to \mathbb{R}\}$ be given by $f_k(x) = \begin{cases} 0 & \text{if} \quad x < k \\ 1 & \text{if} \quad x \geq k \end{cases}$. Show that the sequence $\{f_k\}$ converges to 0 but the convergence is not uniform. (4)

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Question Five

5.1 Let $\{f_k : A \to \mathbb{R}^m\}$ be a sequence of continuous functions, and suppose $f_k \to f$ **uniformly**. Prove that f is continuous. (4)

5.2 (a) State, without proof, the Weierstrass M-test for uniform convergence. (2)

(b) Show that $\sum_{n=1}^{\infty} \frac{(\sin(nx))^2}{n^2}$ converges uniformly. (2)

5.3 (a) State, without proof, the Contraction Mapping Principle. (2)
(b) Give an example of a complete metric space X and a map $T : X \to X$ with $d(T(x), T(y)) \leq d(x, y)$ but having no fixed point. (3)

5.4 (a) Let $f : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$. Define what is meant by f is differentiable at $x_0 \in A$. (2)
(b) Define the Jacobian matrix of the function f in (a) above. (3)
(c) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be given by $f(x, y) = (x^2, x^3y, x^4y^2)$. Compute the Jacobian matrix of f. (2)

5.5 Let $x, y \in R$. Let $U(x, y) = \frac{x^4+y^4}{x}$ and $V(x, y) = \cos x + \sin y$. Find at least one point near which we can solve for x, y in term of U, V. (4)

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Question Six

6.1 Let $x, y \in R$. Let $U(x, y) = e^x \sin y$ and $V(x, y) = e^x \cos y$. Show that $f(x, y) = (U(x, y), V(x, y))$ is locally invertible, but NOT invertible. (4)

6.2 (a) Define a partition of an interval $[a, b]$. (b) Define the upper and the lower sums of a function $f : [a, b] \to [0, \infty)$. (c) Define what is meant by f is Riemann integrable. (5)

6.3 Let Q^c and Q denote the sets of irrational and rational numbers respectively. Let $f : [0, 1] \to \mathbb{R}$ be given by $f(x) = \begin{cases} 1 \text{ if } x \in Q^c \\ 0 \text{ if } x \in Q \end{cases}$ Show that f is NOT Riemann integrable. (4)

6.4 (a) Define the volume of a subset A of $\mathbb{R}^n$. (b) If $A = [3,5] \times [-1,2]$ and $n = 2$, find the volume of A. (c) Define what is meant by a subset A of $\mathbb{R}^n$ has measure zero. (4)

6.5 (a) State, without proof, Lebesgue's Theorem. (b) State a corollary of the Lebesgue Theorem that characterises a set of measure zero in terms of volume. (4)

6.6 Let A be a set and $\mathcal{P}(A)$ the power set of A. Prove that $|A| < |\mathcal{P}(A)|$. (4)

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