UNIVERSITY OF FORT HARE

Advanced Mathematical Methods MAP 511

Honours Degree Examinations

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Time: 3 Hrs

Subject: Applied Mathematics

Marks: 100

This question paper consists of 3 pages

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Instructions

Answer all questions.

Symbols have the usual meanings

Question 1

1.1 Show that an even function can have no sine terms in its Fourier expansion. (4)

1.2 If $f(x)$ is an odd function, show that $a_n = 0$ and $b_n = \frac{2}{L}\int_0^L f(x) sin\frac{n\pi x}{L}dx.$ (6)

1.3 Prove that for $0 \le x \le \pi$, the half Fourier cosine expansion of

$$x(\pi - x) = \frac{\pi^2}{6} - \left(\frac{cos2x}{1^2} + \frac{cos4x}{2^2} + \frac{cos6x}{3^2} + \cdots\right).$$

(8)

1.4 Hence show that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$. (4)

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Question 2

2.1 An important application of the Fourier transforms is the resolution of a finite sinusoidal wave into an infinite one. If $sin\, w_0\, t$ is clipped so that Ncycles remain then,

$$f(t) = \begin{cases} sin\, w_0\, t, & for \quad |t| < \dfrac{N\pi}{w_0}, \\ 0 & elsewhere \end{cases}$$

(a) Use the Fourier sine transform to show that $F(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^{N\pi/w_0} sin\, w_0\, t\, sin(\alpha t)dt.$ (3)

(b) Compute $F(w_0)$. (5)

2.2 If a function $f(t)$ has a period w that is, $f(t + w) = f(t)$ then show that the Laplace transform of $f(t)$ is $\mathcal{L}(f) = \frac{1}{1-e^{-sw}}\int_0^w e^{-st}f(t)dt.$ (7)

2.3 Use the Laplace transforms to solve the following differential equation;

$y''(t) + 16y(t) = 32t$, subject to $y(0) = 3, \quad y'(0) = -2$ (8)

2.4 If $\mathcal{L}^{-1}[F(s)] = f(t)$ and $\mathcal{L}^{-1}[G(s)] = g(t)$ then show that the convolution of $f(t)$ with $g(t)$, i.e. $\mathcal{L}^{-1}[F(s)G(s)] = \int_0^t f(t-\alpha)g(\alpha)d\alpha.$ (10)

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Question 3

3.1 Consider the following linear second order differential equation;

$$y'' + A(x)y' + B(x)y = g(x)$$

subject to the initial conditions $y(a) = y_0, \quad y'(a) = y'_0$, and derive the Volterra's integral equation of the second kind. (10)

P.T.O

3.2 If $f(z)$ is an analytic function everywhere inside and a simple closed curve Cexcept at $z = a$ which is a pole of order n so that

$$f(z) = \frac{a_{-n}}{(z-a)^n} + \frac{a_{-n+1}}{(z-a)^{n-1}} + \cdots + a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$$

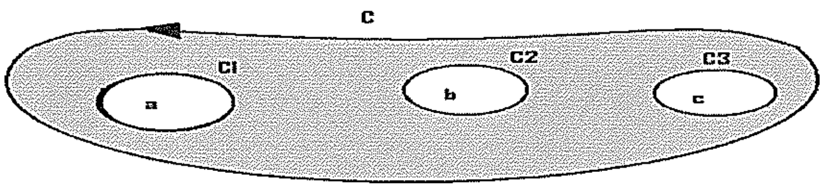
where $a_{-n} \neq 0$, and a_{-1} is a Residue, prove that

3.2.1 $\oint_C f(z)dz = 2\pi i a_{-1}$. (5)

3.2.2 $a_{-1} = \lim_{z \to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \{(z-a)^n f(z)\}$ (5)

3.3 Determine the residue(s) of the following function at its poles; $\frac{z}{(z^2+1)^2}$. (6)

3.4 If $f(z)$ is an analytic function within and on a simple closed curve C, (as shown in the diagram below) except at a number of poles $a,\ b,\ c,\ \ldots$ interior to C,



prove that

$$\oint_C f(z)dz = 2\pi i\{\text{sum of Residues of } f(z) \text{ at poles } a,\ b,\ c,\ \ldots\}.$$ (6)

3.5 In a simple closed curve$C$ enclosing the circles$z = \pm i$, show that

$$\oint_C \frac{ze^{zt}}{(z^2+1)^2} dz = \frac{1}{2} t \sin t.$$ (8)

[30]

THE END

P.T.O