University of Fort Hare

MAT 311

Degree Examinations: June 2017

Subject: Mathematics 3
Paper: Abstract Algebra

Time: 3 Hours | Marks: 100 | Subminimum: 40

This question paper consists of 4 pages

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Instructions

Attempt **NO MORE** than **FIVE (5)** questions. Symbols used have the usual meanings. G is a group unless stated otherwise.

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Question One

1.1 Let a, b, $c \in \mathbb{Z}$ with gcd $(a, c) = 1$. If $a \mid bc$, show that $a \mid b$. (3)

1.2 Give an example of a group of order 4 having the property that each of its non-identity elements has order 2. Further, argue that such a group cannot be cyclic. (3)

1.3 Show that the group $G = \{z \in C \,:\, z^n = 1,\ n \in \mathbb{N}\}$, of all the n-th roots of 1, is cyclic. (3)

1.4 Define a binary operation $\star$ on Q^+ (the set of all positive rational numbers) by $a \star b = \frac{2}{3}ab$ for all a, $b \in Q^+$. Then $(Q^+, \star)$ is a group. Find the identity element and the inverse of an element x in this group. (3)

1.5 Let $S = \mathbb{R} - \{-1\}$. Define $\star$ on S by $a \star b = a + b + ab$. (a) Show that $\star$ gives a binary operation on S; (b) Show that S with $\star$ is a group; (c) Find the solution of the equation $2 \star x \star 3 = 9$ in S. (8)

1.6 Let $\star$ be a binary operation on a group G. If for every $x \in G$, $x \star x = e$, show that G is abelian. (3)

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Question Two

2.1 Find all the subgroups of the group $\mathbb{Z}_{18}$ and then draw a lattice diagram for the subgroups . (5)

2.2 Prove that a subgroup of a cyclic group is cyclic. (5)

2.3 Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 5 & 4 \end{pmatrix}$ in S_6.
Find (a) σ^{-1}; (b) the order of σ; (c) σ^{231}. (6)

2.4 Prove that in the group S_n, $n \geq 2$, the number of odd permutations is equal to the number of even permutations. (5)

2.5 Let G be a group of order 55. Argue that every proper subgroup of G is cyclic. (3)

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Question Three

3.1 Show that the group $\mathbb{Z}_4 \times \mathbb{Z}_4$ has no element of order 16 by examining the order of each element of the group. (3)

3.2 State and prove Lagrange's Theorem. Deduce that every group of prime order p is cyclic. (6)

3.3 Find the number of left cosets of $H = < (1,4) >$ in the group $\mathbb{Z}_{15} \times \mathbb{Z}_{24}$, **without** enumerating the elements of H. (3)

3.4 (a) Let G be a group. Define a normal subgroup of G.
(b) Let $f : G \to H$ be a homomorphism from a group G into a group H. Define the kernel of f, and then prove that the kernel of f is a normal subgroup of G. (5)

3.5 Prove that a group homomorphism is one-to-one if and only if $ker\, f = \{e\}$. (3)

3.6 Show that if a finite group G has exactly one subgroup H of a given order n, then H is normal in G. (2)

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Question Four

4.1 Compute the order of the coset $(2,3) + < (2,2) >$ in the group $(\mathbb{Z}_6 \times \mathbb{Z}_8) / < (2,2) >$. (4)

4.2 Compute the group $(\mathbb{Z}_6 \times \mathbb{Z}_{12}) / < (2,3) >$ (i.e. find the simplest group to which it is isomorphic). (3)

4.3 Let $(S = \mathbb{R} - \{-1\}, \star)$ be the group whose binary operation is given by $a \star b = a + b + ab$ for $a,\ b \in S$. Show that $(S = \mathbb{R} - \{-1\}, \star)$ is isomorphic to the group $(\mathbb{R} - \{0\}, \times)$. (5)

4.4 (a) Define the center $Z(G)$ of a group G . (2)
(b) Show that (i) $Z(G)$ is a subgroup of G and (ii) $Z(G)$ is normal in G. (4)

4.5 (a) Define the commutator subgroup G' of a group G . (2)
(b) Show that the commutator subgroup G' of G is normal in G. (3)

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Question Five

5.1 Define a (i) ring, (ii) integral domain, (iii) division ring. (4)

5.2 Define $f : \mathbb{Z} \to 2\mathbb{Z}$ by $f(n) = 2n$. Is f a ring homomorphism? A group homomorphism? Explain. (2)

5.3 Show that the ring $\mathbb{Z} \times \mathbb{Z}$ is not an integral domain. (2)

5.4 Find all zero divisors in the ring $\mathbb{Z}_{24}$. (3)

5.5 Prove that every finite integral domain is a field. (4)

5.6 (a) Define the characteristic of a ring R.
(b) Find the characteristic of the ring $\mathbb{Z}_{21} \times \mathbb{Z}_{14}$. (2)

5.7 Let R be a ring with unity and N an ideal of R containing a unit. Prove that $N = R$. (4)

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Question Six

6.1 (a) Discuss an example showing that a factor ring of an integral domain may have zero divisors. (2)
(b) Discuss an example showing that a factor ring of a ring with zero divisors may be an integral domain. (3)
(c) Discuss a subring of the ring $\mathbb{Z} \times \mathbb{Z}$ that is not an ideal of $\mathbb{Z} \times \mathbb{Z}$. (3)

6.2 Prove that if F is a field, then every ideal in the ring $F[x]$ is principal. (6)

6.3 (a) Find a maximal ideal of $\mathbb{Z} \times \mathbb{Z}$, giving reasons. (2)
(b) Find a prime ideal of $\mathbb{Z} \times \mathbb{Z}$ that is not maximal, giving reasons. (2)

6.4 Is $Q[x]/ < x^2 + 5x + 3 >$ a field? Explain. (Q is the field of rational numbers). (2)

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END