UNIVERSITY OF FORT HARE

ADVANCED NUMERICAL DIFFERENTIATION AND INTEGRATION

MAP 312

JUNE EXAMINATIONS

Time: **3 hours**

Subject: **Applied Mathematics**

Marks: **175 (180 available)**

This paper consists of 6 pages including the cover page.

Examiner: **Dr. S. J. Childs**

External examiner: **Dr. K. Dukuza**

INSTRUCTIONS

All questions may be answered.

Show your working.

Sloppy work will be penalised.

1. Suppose that $f(\xi)$, in

$$\int_{\alpha}^{\beta} w(\xi)f(\xi)d\xi,$$

is a polynomial of degree no greater than $2n-1$, that $P_k(\xi)$ is a polynomial of degree k and that $\{P_0(\xi), P_1(\xi), \ldots, P_n(\xi)\}$ constitutes a set of polynomials which are mutually orthogonal with respect to the weighting function, $w(\xi)$, on $[\alpha, \beta]$. That is,

$$\int_{\alpha}^{\beta} w(\xi)P_k(\xi)P_l(\xi)d\xi \begin{cases} = 0, \ k \neq l \\ \neq 0, \ k = l \end{cases}.$$

   (a) Decompose the former integrand, $w(\xi)f(\xi)$, into two terms which will facilitate proof of the Gaussian method of quadrature, and indicate the maximum degree of any polynomials that arise. (13 marks)

   (b) Show that the integral of one of these terms vanishes. (13 marks)

   (c) Show that the error associated with the Lagrange interpolation of $f(\xi)$, in $\int_{\alpha}^{\beta} w(\xi)f(\xi)d\xi$, is zero if the abscissae are the n roots of $P_n(\xi)$ (you may assume that $P_k(\xi)$ has k distinct roots on $[\alpha, \beta]$). (20 marks)

   (d) Deduce the formula for the weights, c_i, in the Gaussian quadrature method from your answer above. (5 marks)

2. Some abscissae and weights for Gauss-Legendre integration are tabulated on page 6. Obtain exact answers to the following using the most efficient Gaussian integration:

(a)

$$\int_{-3}^{5} x^4 + 4x + 1 \, dx$$

(16 marks)

(b)

$$\int_{-2}^{6} (x+1)^5 dx$$

(16 marks)

3. Show that the initial value problem (IVP),

$$\frac{dy}{dt} = f(t, y) = 1 + \sin(t^2 y), \qquad y(0) = 0, \qquad \frac{17\pi}{6} \leq t \leq 3\pi,$$

has a unique solution. (11 marks)

4. Euler's method was the first, as well as the most basic numerical method you learned about for solving IVP's. Although highly tempting to the uninitiated, it should be avoided in practice (except in instances where the most elementary, qualitative understanding of the solution is sought, or where the most rudimentary, quantitative values are sought). Despite this it retains an eminent status in undergraduate numerical analysis. Explain ALL of the above in detail. (18 marks)

5. Consider a generalised multistep method

$$\begin{aligned} w_{i+1} &= a_m w_i + a_{m-1} w_{i-1} \ldots + a_0 w_{i-m} \\ &\quad + h\left[b_{m+1} f(t_{i+1}, w_{i+1}) + b_m f(t_i, w_i) + \ldots + b_0 f(t_{i-m}, w_{i-m})\right] \end{aligned}$$

to be used for the points $i = m, m+1, \ldots, n-1$, given $w_0 = \alpha_0$,

$w_1 = \alpha_1, \ldots, w_m = \alpha_m$, $h = \frac{(b-a)}{n}$ and $t_i = a + ih$.

(a) For what values of b_{m+1} is the method implicit? (2 marks)

(b) For what value of b_{m+1} is the method explicit? (2 marks)

(c) Which method would you prefer to use and why? (2 marks)

(d) In practice, why might you not be able to use that method? (2 marks)

(e) How does the use of a predictor-corrector method circumvent these problems? (2 marks)

6. The exact formula for the Adams-Bashforth, four-step, $O(h^4)$-accurate method is

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + \frac{h}{24}\left[55 f(t_i, w_i) - 59 f(t_{i-1}, w_{i-1}) + 37 f(t_{i-2}, w_{i-2})\right. \\ &\quad \left. - 9 f(t_{i-3}, w_{i-3})\right] + \frac{251}{720} h^5 f^5(\xi_{i+1}), \end{aligned}$$

for $t_{i-3} < \xi_{i+1} < t_{i+1}$. The exact formula for the Adams-Moulton, three-step, $O(h^4)$-

accurate method is

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + \frac{h}{24}\left[9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1})\right. \\ &\quad \left. + f(t_{i-2}, w_{i-2})\right] - \frac{19}{720}h^5 f^5(\tilde{\xi}_{i+1}), \end{aligned}$$

for $t_{i-2} < \tilde{\xi}_{i+1} < t_{i+1}$. These two multistep methods are suitably matched for use in a predictor-corrector algorithm. Since predictor-corrector methods give rise to two estimates at each step, they lend themselves naturally to an error control procedure.

(a) Formulate the error, at each step in the algorithm, in terms of the difference between the solutions of the predictor stage, $w_{i+1}^{(0)}$, and the corrector stage, $w_{i+1}^{(1)}$, as well as the step size, h. (39 marks)

(b) Show that setting a tolerance, ϵ, on this error enables one to formulate an expression for a scaling factor, q, with which to vary the step size, h. (19 marks)

Number of abscissae used (n)	**Abscissa** (ξ_i)	**Weights** (c_i)	**Maximum degree of polynomial for exact integration** ($2n-1$)	**Error formula**
1	$\xi_1 = 0$	$c_1 = 2$	1	$\frac{1}{6}\frac{d^2y}{d\xi^2}$
2	$\xi_1 = -\frac{1}{\sqrt{3}}$ $\xi_2 = \frac{1}{\sqrt{3}}$	$c_1 = 1$ $c_2 = 1$	3	$\approx 0.7\times 10^{-2}\frac{d^4y}{d\xi^4}$
3	$\xi_1 = -\sqrt{\frac{3}{5}}$ $\xi_2 = 0$ $\xi_3 = \sqrt{\frac{3}{5}}$	$c_1 = \frac{5}{9}$ $c_2 = \frac{8}{9}$ $c_3 = \frac{5}{9}$	5	$\approx 0.6\times 10^{-4}\frac{d^6y}{d\xi^6}$
4	$\xi_1 = -\sqrt{\frac{3+2\sqrt{\frac{6}{5}}}{7}}$ $\xi_2 = -\sqrt{\frac{3-2\sqrt{\frac{6}{5}}}{7}}$ $\xi_3 = \sqrt{\frac{3-2\sqrt{\frac{6}{5}}}{7}}$ $\xi_4 = \sqrt{\frac{3+2\sqrt{\frac{6}{5}}}{7}}$	$c_1 = (\frac{1}{2} - \frac{1}{6\sqrt{\frac{6}{5}}})$ $c_2 = (\frac{1}{2} + \frac{1}{6\sqrt{\frac{6}{5}}})$ $c_3 = (\frac{1}{2} + \frac{1}{6\sqrt{\frac{6}{5}}})$ $c_4 = (\frac{1}{2} - \frac{1}{6\sqrt{\frac{6}{5}}})$	7	$\approx 0.3\times 10^{-6}\frac{d^8y}{d\xi^8}$
5	$\xi_1 = -\frac{1}{3}\sqrt{5+4\sqrt{\frac{5}{14}}}$ $\xi_2 = -\frac{1}{3}\sqrt{5-4\sqrt{\frac{5}{14}}}$ $\xi_3 = 0$ $\xi_4 = \frac{1}{3}\sqrt{5-4\sqrt{\frac{5}{14}}}$ $\xi_5 = \frac{1}{3}\sqrt{5+4\sqrt{\frac{5}{14}}}$	$c_1 = (\frac{161}{450} - \frac{13}{180\sqrt{\frac{5}{14}}})$ $c_2 = (\frac{161}{450} + \frac{13}{180\sqrt{\frac{5}{14}}})$ $c_3 = \frac{128}{225}$ $c_4 = (\frac{161}{450} + \frac{13}{180\sqrt{\frac{5}{14}}})$ $c_5 = (\frac{161}{450} - \frac{13}{180\sqrt{\frac{5}{14}}})$	9	$\approx 0.8\times 10^{-9}\frac{d^{10}y}{d\xi^{10}}$