UNIVERSITY OF FORT HARE

ADVANCED NUMERICAL DIFFERENTIATION AND INTEGRATION

MAQ 522

NOVEMBER EXAMINATIONS

Time: 3 hours

Subject: Applied Mathematics

Marks: 180

This paper consists of four pages including the cover page.

Examiner: **Dr. S. J. Childs** External examiner: **Prof. G. Lubczonok**

INSTRUCTIONS

All questions may be answered.

Show your working.

Sloppy work will be penalised.

1. (a) Derive the second order Runge-Kutta method. (20 marks)

   (b) Use the version that is the modified Euler's method to obtain f(0.05), by solving $\frac{df}{dx} = 3e^{2x}$, with an initial value of $f(0) = \frac{1}{5}$ and a step length of $h = 0.05$. (12 marks)

2. Suppose that

$$w_{i+1} = w_i + h\phi(t_i, w_i, h), \qquad w_0 = \alpha$$

and

$$\tilde{w}_{i+1} = \tilde{w}_i + h\tilde{\phi}(t_i, \tilde{w}_i, h), \qquad \tilde{w}_0 = \alpha$$

are general methods for solving an I.V.P., that the first has an $O(h^n)$ local truncation error in ϕ and that the second has an $O(h^{n+1})$ local truncation error in $\tilde{\phi}$, then:

   (a) Show that $\tau_{i+1} \approx \frac{1}{h}(\tilde{w}_{i+1} - w_{i+1})$ to first order. (22 marks)

   (b) Formulate $\tau_{i+1}(qh)$ in terms of $\tau_{i+1}(h)$, the original, $O(h^n)$ local truncation error ($q > 0$). (5 marks)

   (c) Re-write this $\tau_{i+1}(qh)$ in terms of $(\tilde{w}_{i+1} - w_{i+1})$. (1 mark)

   (d) If one sets a tolerance, ϵ, on this value, then $|\tau_{i+1}| \leq \epsilon$. Formulate an expression for the q that this tolerance dictates. (8 marks)

3. Based on your answer to the previous question, outline the steps you would implement

in a Runge-Kutta-Fehlberg, RKF4-5 algorithm which implements

$$w_{i+1} = w_i + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 - \frac{1}{5}k_5$$

and

$$\tilde{w}_{i+1} = \tilde{w}_i + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6,$$

in which

$$\begin{aligned}
k_1 &= hf(t_i, w_i) \\
k_2 &= hf(t_i + \frac{1}{4}h, w_i + \frac{k_1}{4}) \\
k_3 &= hf(t_i + \frac{3}{8}h, w_i + \frac{3}{32}k_1 + \frac{9}{32}k_2) \\
k_4 &= hf(t_i + \frac{12}{13}h, w_i + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3) \\
k_5 &= hf(t_{i+1}, w_i + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4) \\
k_6 &= hf(t_i + \frac{1}{2}h, w_i - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2363}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5).
\end{aligned}$$

(25 marks)

4. The exact formula for the Adams-Bashforth, four-step, $O(h^4)$-accurate method is

$$\begin{aligned}
y(t_{i+1}) &= y(t_i) + \frac{h}{24}\left[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2})\right. \\
&\quad \left. -9f(t_{i-3}, w_{i-3})\right] + \frac{251}{720}h^5 f^5(\xi_{i+1}),
\end{aligned}$$

for $t_{i-3} < \xi_{i+1} < t_{i+1}$. The exact formula for the Adams-Moulton, three-step, $O(h^4)$-accurate method is

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + \frac{h}{24}\left[9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1})\right. \\ &\quad \left. + f(t_{i-2}, w_{i-2})\right] - \frac{19}{720}h^5 f^5(\tilde{\xi}_{i+1}), \end{aligned}$$

for $t_{i-2} < \tilde{\xi}_{i+1} < t_{i+1}$. These two multistep methods are suitably matched for use in a predictor-corrector algorithm. Since predictor-corrector methods give rise to two estimates at each step, they lend themselves naturally to an error control procedure.

(a) Formulate the error, at each step in the algorithm, in terms of the difference between the solutions of the predictor stage, $w_{i+1}^{(0)}$, and the corrector stage, $w_{i+1}^{(1)}$, as well as the step size, h. (39 marks)

(b) Show that setting a tolerance, ϵ, on this error enables one to formulate an expression for a scaling factor, q, with which to vary the step size, h. (19 marks)

5. Based on the information in the previous question, outline the steps you would implement in an Adams, $O(h^4)$-accurate, variable-step-size, predictor-corrector algorithm. (31 marks)