UNIVERSITY OF FORT HARE

NUMERICAL ANALYSIS

MAP 211

JUNE EXAMINATIONS

Time: 3 hours

Subject: Applied Mathematics

Marks: 130 (140 available)

This paper consists of 4 pages including the cover page.

Examiner: **Dr. S. J. Childs**

INSTRUCTIONS

All questions may be answered.

Show your working.

Sloppy work will be penalised.

1. In this course you discovered that numerical differentiation is **not** unconditionally stable, whereas integration is.

   (a) Suppose the precision of your machine results in a roundoff error of $e(x \pm h)$. Then $f(x \pm h) = \tilde{f}(x \pm h) + e(x \pm h)$. Derive a formula for the optimal h when implementing a central difference approximation of the derivative,

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{6} f'''(\xi), \qquad x - h \leq \xi \leq x + h.$$

(14 marks)

   (b) Suppose a central difference is to be used to approximate the derivative of $f(x) = \sin 2x$ on the interval $[0.7, 0.9]$ and a tolerance of 0.00002 is set on roundoff error. Find the optimal h. (6 marks)

2. If one adds the Taylor series expansions of $f(x+h)$ and $f(x-h)$, one obtains

$$f^{\mathrm{II}}(x) = \frac{1}{h^2}\left[f(x+h) - 2f(x) + f(x-h)\right] - \frac{h^2}{12} f^{\mathrm{IV}}(x) - \frac{h^4}{360} f^{\mathrm{VI}}(x) + \ldots$$

   which has an $O(h^2)$ error. Use Richardson extrapolation on this equation to obtain an expression for $f^{\mathrm{II}}(x)$ with an $O(h^4)$ error. (18 marks)

3. Solve the equation

$$\frac{df}{dx} = \sin x$$

   using an initial value of $f(1) = 2$ and a step length of $h = 0.02$, to obtain $f(1.04)$ using

(a) Euler's method. (13 marks)

(b) The midpoint rule. (7 marks)

(c) The method derived from the backward difference. (13 marks)

4. The Lagrange interpolating polynomial led you to discover an important generalisation; namely that all the basic, closed Newton-Cotes schemes, for the integration of a function, $f(x)$, over $[a, b]$, can be formulated as

$$\sum_{i=0}^{n} c_i f(x_i),$$

in which n is the number of increments, of size h, and

$$c_i = \frac{h(-1)^{n-i}}{i!(n-i)!} \int_0^n t(t-1)\cdots(t-i+1)(t-i-1)\cdots(t-n)\, dt$$

is the formula for the weights. For the basic, $n = 3$ rule:

(a) Work out the weights, c_i. (20 marks)

(b) Formulate the resulting basic rule. (5 marks)

(c) Name this basic rule. (1 mark)

5. For the purposes of Romberg integration, the composite trapezoidal rule can be stated as

$$\begin{aligned} \int_a^b f(x)dx &= \frac{h_k}{2}\left[f(a) + f(b) + 2\sum_{j=1}^{2^{(k-1)}-1} f\left(a + jh_k\right)\right] - \frac{(b-a)h_k^2}{12} f^{II}(\xi) \\ &= R_{k,1} - \frac{(b-a)h_k^2}{12} f^{II}(\xi), \qquad a \leq \xi \leq b, \qquad (1) \end{aligned}$$

for $m_k = 2^{k-1}$ subdivisions of the interval $[a, b]$.

(a) Write out the basic trapezoidal rule, $R_{1,1}$, then, for the successive subdivisions $k = 2, 3$ and 4 of the interval, derive $R_{k,1}$, as a function of $R_{k-1,1}$ and h_{k-1}. (8 marks)

(b) Based on the pattern you observe, write down a general formula for $R_{k,1}$ in terms of $R_{k-1,1}$ and h_{k-1}. (8 marks)

(c) The error term in equation (1), above, can be rewritten so that equation (1) becomes,

$$\int_a^b f(x)dx \quad = \quad R_{k,1} - \frac{h_k^2}{12}\left[f^I(b) - f^I(a)\right] + \frac{(b-a)h_k^4}{720} f^{IV}(\xi_2) \, .$$

Use Richardson extrapolation to obtain $R_{k,2}$, in terms of $R_{k,1}$, $R_{k-1,1}$ and h_k, with an $O(h^4)$ error. (10 marks)

(d) Which Newton-Cotes formula error is this $O(h^4)$ error identical to? (1 mark)

(e) What is $R_{k,3}$ (has an $O(h^6)$ error)? (5 marks)

(f) Generalise to obtain a formula for $R_{k,j}$, which will form a second sequence. (9 marks)

(g) The $R_{2,2}$ error is identical to the error of what other method that you learned about? (2 marks)