UNIVERSITY OF FORT HARE

STM 322

DEGREE EXAMINATIONS
JAN/FEB 2019

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TIME : 3 HOURS

SUBJECT : ADVANCED MATHEMATICAL STATISTICS A2

MARKS : 100

This paper consists of 4 printed pages including the cover page

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Instructions

Answer all questions

Statistical Tables will be provided

QUESTION ONE [42 POINTS]

1.1 State the result of finding a complete sufficient statistic $T(\mathbf{X})$, for a parameter θ, from the exponential class of distributions. **[5]**

1.2 Consider a random sample $X_1, X_2, \ldots, X_n$ from an absolutely continuous distribution on $\mathbb{R}$ with density

$$f(x;\theta) = \frac{\theta}{(1+x)^{\theta+1}}, \quad x \geq 0$$

(a) Show that $f(x;\theta)$ belongs to the exponential family of probability density functions (*you are required to identify all components*) and give $T(\mathbf{X})$, a complete sufficient statistic for θ. **[4]**

(b) Find a function of $\hat{\theta}$, the mle of θ which is an unbiased estimator of θ.
Hint: $\log(1+x)$ has a familiar distribution. **[10]**

(c) Compute the Cramer-Rao Lower Bound for unbiased estimators of θ. **[3]**

1.3 Let

$$f(x;\theta) = \begin{cases} e^{-(x-\theta)}, & x > \theta \\ 0, & x \leq \theta \end{cases}$$

Let $y_1 < y_2 < y_3$ be ordered values of x_1, x_2, x_3.

(a) Show that the first order statistic, y_1 is sufficient for θ. **[6]**

(b) Show that the distribution of $Y_1 - \theta$ is exponential of parameter n. Your client suggests the following possibilities as estimators of θ:
$T_1 = x_3 - 1$
$T_2 = y_1$
$T_3 = \frac{1}{3}(x_1 + x_2 + x_3) - 1$
How would you advise him?
General theorems used should be clearly stated, but need not be proved. **[8]**

(d) Prove for $f(x;\theta)$ that the information provided by a sufficient statistic $T(\mathrm{X})$ is the same as that in the sample $\mathbf{X}$. **[6]**

QUESTION TWO [40 POINTS]

2.1 Write a short account of the standard procedure used by statisticians for hypothesis testing. Your account should explain, in particular, why the null hypothesis is considered differently from the alternative and also say what is meant by a best critical region. **[5]**

2.2 There is a single observation of a random variable X which has a PDF $f(x)$.

(a) Construct the best test of size 0.05 for the null hypothesis **[4]**

$$H_0 : f(x) = \tfrac{1}{2}, \quad (-1 \leq x \leq 1)$$

against the alternative hypothesis

$$H_1 : f(x) = \tfrac{3}{4}(1 - x^2), \quad (-1 \leq x \leq 1)$$

(b) Calculate the power of your test. **[5]**

2.3 The lifetime of certain electronic components produced by machine X may be assumed to follow the exponential PDF

$$f(x;\theta) = \frac{1}{\theta}\exp(-x/\theta)$$

where x is the sample value of X.

(a) Let $X_1, X_2, \cdots, X_n$ be a random sample from this PDF. Quoting carefully the NP Lemma, find the form of the most powerful test of size 0.05 of

$H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$

where θ_0 and θ_1 are given, and $\theta_0 < \theta_1$. **[5]**

(b) Defining the function

$$G_n(u) = \int_0^u e^{-x} \frac{x^{n-1}}{(n-1)!} dx,$$

show that this test has power

$$1-G_n\left[\frac{\theta_0}{\theta_1}G_n^{-1}(1-\alpha)\right],$$

where $\alpha = 0.05$. **[5]**

2.4 Let $Y_1, Y_2, \cdots Y_m$ and $X_1, X_2, \ldots, X_n$ be independent samples, respectively, from $N(\mu_1, \sigma^2)$ and the $N(\mu_2, \sigma^2)$-distributions. Here the parameters μ_1, μ_2, and σ^2 are all unknown. Explain carefully how you would test the hypothesis $H_0: \mu_1 = \mu_2$ against $H_1: \mu_1 \neq \mu_2$. **[16]**

QUESTION THREE [18 POINTS]

3.1 Let $X_1, X_2, \ldots, X_n$ be a sample from a uniform distribution on $[0, \theta]$, where $\theta \in [1, 2]$ is an unknown parameter. Find an unbiased estimator for θ of variance less than $1/10$. **[8]**

3.2 Consider the simple linear regression equation

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i \qquad i = 1, 2, \ldots, n$$

where Y_i are independent and normally distributed with mean μ and unit variance.

Write down the information matrix for β_0 and β_1. **[10]**