University of Fort Hare

MAT 311

Supplementary Examinations: November 2018

Subject: Mathematics 3
Paper: Abstract Algebra

Time: 3 Hours | Marks: 100 | Subminimum: 40

This question paper consists of 4 pages

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Instructions

Attempt **NO MORE** than **FIVE (5)** questions. Symbols used have the usual meanings. G is a group and e is the identity element unless stated otherwise.

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Question One

1.1 Let a, b, $c \in Z$ with gcd $(a, c) = 1$. If $a \mid bc$, show that $a \mid b$. (3)

1.2 Define a binary operation $\star$ on Q^+ (the set of all positive rational numbers) by $a \star b = 5ab$ for all a, $b \in Q^+$. Prove that $(Q^+, \star)$ is a group. (4)

1.3 Let $S = \mathbb{R} - \{-1\}$. Define $\star$ on S by $a \star b = a + b + ab$.
(a) Show that $\star$ gives a binary operation on S.
(b) Show that S with $\star$ is a group.
(c) Find the solution of the equation $6 \star x \star (-2) = 13$ in S. (8)

1.4 Find all the subgroups of the group Z_{18} and then draw a lattice diagram for the subgroups . (5)

1.5 Prove that a subgroup of a cyclic group is cyclic. (5)

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Question Two

2.1 Give an example, with explanation, of a non-abelian group all of whose proper subgroups are cyclic. (Ensure that you show that your group is non-abelian and that indeed all its proper subgroups are cyclic). (5)

2.2 Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 6 & 4 & 5 \end{pmatrix}$ be a permutation in S_6.
Find (a) σ^{-1}; (b) the order of σ; (c) σ^{323}; (5)

2.3 (a) Express the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 5 & 1 & 8 & 7 & 2 \end{pmatrix}$ as (i) a product of disjoint cycles; (ii) a product of transpositions .
(b) Is σ even or odd? Explain. (4)

2.4 (a) Let G be a group of order 21. Show that every proper subgroup of G is cyclic. (3)

(b) Compute all the left cosets of the subgroup $< 5 >$ of $\mathbb{Z}_{15}$ (2)

2.5 Argue convincingly that the group $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic by exhibiting an element of the group that generates all the others. (3)

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Question Three

3.1 (a) Let G be a group. Define a normal subgroup of G.

(b) Let $f : G \to H$ be a homomorphism from a group G into a group H. Define the kernel of f, and then prove that the kernel of f is a normal subgroup of G. (4)

3.2 Show that if a finite group G has exactly one subgroup H of a given order n, then H is normal in G. (3)

3.3 Prove that a group homomorphism f is one-to-one if and only if $ker\, f = \{e\}$. (3)

3.4 (a) Define the center of a group G . (2)

(b) Define the commutator subgroup G' of a group G . (2)

(c) Show that if G is a non-abelian group, then $G/Z(G)$ is non-cyclic where $Z(G)$ is the center of G. (4)

(d) Show that the commutator subgroup G' of G is normal in G. (3)

3.5 Compute the order of the coset $(1, 2)+ < (1, 0) >$ in the group $(\mathbb{Z}_4 \times \mathbb{Z}_6)/ < (1, 0) >$. (3)

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Question Four

4.1 Find a simpler group to which the group $(\mathbb{Z}_4 \times \mathbb{Z}_6)/ < (0,3) >$ is isomorphic. Motivate your answer. (3)

4.2 Let $< S = \mathbb{R} - \{-1\}, \star) >$ be the group whose binary operation is given by $a \star b = a + b + ab$ for $a, b \in S$, as in Question 1.3. Show that $< S = \mathbb{R} - \{-1\}, \star) >$ is isomorphic to the group $(\mathbb{R} - \{0\}, \times)$. (5)

4.3 Define (i) a ring, (ii) an integral domain, (iii) a division ring. (4)

4.4 Show that $M_3(Z)$, the ring of all 3×3 matrices with integer entries, is not an integral domain. (3)

4.5 Prove that every finite integral domain is a field. (4)

4.6 Show that $f(x) = x^3 + 2x + 3$ is not irreducible in $\mathbb{Z}_5[x]$. Then express $f(x)$ as a product of irreducible polynomials in $\mathbb{Z}_5[x]$. (4)

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Question Five

5.1 (a) Prove that in the ring $\mathbb{Z}_n$, the zero divisors are precisely those elements that are not relatively prime to n. Hence
(b) find all zero divisors in the ring $\mathbb{Z}_{12}$. (5)

5.2 (a) Define the characteristic of a ring R.
(b) Find the characteristic of the ring $\mathbb{Z}_{12} \times \mathbb{Z}_{10}$. (3)

5.3 Define $f : \mathbb{Z}_p \to \mathbb{Z}/p\mathbb{Z}$, for a prime p, by $f(m) = m + p\mathbb{Z}$. Show that f is a ring isomorphism. (5)

5.4 (a) Define an ideal of a ring R. (2)
(b) Find a subring of the ring $Z \times Z$ that is not an ideal of $Z \times Z$. (2)

5.5 Let R be a ring with unity and N an ideal of R containing a unit. Prove that $N = R$. (4)

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Question Six

6.1 (a) Define a prime ideal of a commutative ring R. (2)
(b) Find a prime ideal of $\mathbb{Z} \times \mathbb{Z}$ that is not maximal, giving reasons. (2)
(c) Find a non-trivial proper ideal of $\mathbb{Z} \times \mathbb{Z}$ that is not prime, giving reasons. (2)

6.2 Prove that if F is a field, then every ideal in the ring $F[x]$ is principal. (6)

6.3 (a) Discuss an example that shows that a factor ring of an integral domain may have zero divisors. (2)
(b) Discuss an example that shows that a factor ring of a ring with zero divisors may be an integral domain. (2)

6.4 Show that each homomorphism of a field is either one-to-one or maps everything onto 0, where 0 is the additive identity element. (4)

6.5 Find a maximal ideal of $\mathbb{Z} \times \mathbb{Z}$, giving reasons. (2)

6.6 Is $Q[x]/ < x^2 - 6x + 5 >$ a field? Explain. (Q is the field of all rational numbers). (2)

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