UNIVERSITY OF FORT HARE

Special and Orthogonal Functions
MAP 311

Degree Examinations

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Time: 3 Hrs

Subject: Applied Mathematics 3

Marks: 100

This question paper consists of 3 pages

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Instructions

Answer **ALL 4** questions.
Symbols have the usual meanings

Question 1

1.1 State the Bessel's function of the first kind and use it to show that

$$\int_0^{\pi/2} J_1(x cos\theta) = \frac{1-cosx}{x}. \quad (10)$$

1.2 The modified Bessel function of the first kind is defined as

$$I_n(x) = \sum_{r=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2r+n}}{r!\,\Gamma(r+n+1)}$$

If n is an integer, prove that $-I_n(x) = I_n$. (5)

1.3 Prove that, $J'_n(x)J_{-n}(x) - J'_{-n}(x)J_n(x) = \frac{2\sin(n\pi)}{\pi x}$. (10)

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Question 2

2.1 Let $P_n(x)$ be the Legendre polynomials of degree, n which satisfy the differential equation, (15)

$$(1-x^2)y'' - 2xy' + n(n+1)y.$$

(i) State the generating function for Legendre polynomials

(ii) Show that $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

(iii) Prove that

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0 & if\ m \neq n \\ \frac{2}{2n+1} & if\ m = n \end{cases}$$

2.2 Expand the function (5)

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & -1 < x < 0 \end{cases}$$

in a series of Legendre polynomials in the form, $\sum_{n=0}^{\infty} A_n P_n(x)$, where$A_n = \frac{2m+1}{2}\int_{-1}^{1} P_n(x)f(x)dx$.

2.3 Show that $\sum_{n=0}^{\infty} P_n(cos\theta) = \frac{1}{2}cosec\left(\frac{\theta}{2}\right)$. (5)

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Question 3

3.1 The Hermite polynomials $H_n(x)$, have the following generating function,

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!}$$

3.1.1 Show that $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$, (5)

3.1.2 Prove that (10)

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)dx = \begin{cases} 0 & if\ m \neq n \\ 2^n n! \sqrt{\pi} & if\ m = n \end{cases}$$

3.2 (i) Prove that the Laguerre polynomials, $L_n(x)$ are orthogonal in the interval $(0, \infty)$ with respect to the weight function e^{-x}. (ii) Hence or otherwise show that $\int_0^\infty e^{-x} {L_n}^2(x)dx = (n!)^2$. (10)

[25]

Question 4

4.1 Consider the following system

$$x^2 y'' + 3xy' + \lambda y = 0,\ 1 \le x \le e,\ y(1) = 0,\ \ y(e) = 0$$

4.1.1 Show that this is a Sturm Lioville system, (2)

4.1.2 Find the eigenvalues and eigenfunctions of the system. (5)

4.1.3 Find the corresponding orthonormal functions. (3)

4.2 Show that the distinct Eigen-values λ_n and λ_m corresponding to the Eigen-functions $y_m(x)$ and $y_n(x)$ of a Sturm-Lioville system are ***real***. (7)

4.3 (i) Find the Green's function and (ii) solution of the following the boundary-value problem below,

$$y'' - 2y' + 2y = x,\ 0 \le x \le \tfrac{\pi}{2},\ y(0) = y(\pi/2) = 0. \quad (8)$$

[25]

THE END