



University of Fort Hare

MAT 224

Supplementary Examinations: January 2019

Subject: Mathematics 2

Paper: Real Analysis

Time: 3 Hours

Marks: 100

Subminimum: 40

This question paper consists of 4 pages

Internal examiner(s)

External examiner(s)

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Instructions

Answer NO more than FIVE (5) questions. Symbols used have the usual meanings.

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Question One

- 1.1 Let A ; B and C be sets. Prove that (a) $A - (B \cap C) = (A - B) \cup (A - C)$;
(b) $A \subseteq B$ if and only if $A \cap B^c = \emptyset$, where $A^c = X - A$. (6)
- 1.2 If $A, B \subseteq X$, we write B^c for $X - B$ and A^c for $X - A$. Prove that
 $A \subseteq B$ iff $B \cup A^c = X$. (3)
- 1.3 (a) Define what is meant by “set A is equivalent (equipotent) to set B ”. (1)
(b) Show (from definition) that the interval $(0, 1)$ is equivalent to the set
 \mathbb{R}^+ of all positive real numbers. (4)
- 1.4 (a) Define what is meant by a “countably infinite set” and hence define
a countable set. (2)
(b) Prove that the set Q of all rational numbers is countable. (4)
(c) Deduce that the set Q^c of all irrational numbers is uncountable. (2)
- 1.5 Let $f : A \rightarrow B$ be an injective function and assume B is countable.
Prove that A is countable. (2)
- [24]

Question Two

- 2.1 (a) Let S be a bounded set. Define what is meant by (i) the supremum
of S and (ii) the infimum of S . (3)
(b) Let $S = \{x : 2x^2 - 5x \geq 3\}$. Find (i) $\sup S$ and $\inf S$, if they exist. (3)

- 2.2 Prove the following: Let $S \subseteq \mathbb{R}$. A real number a is the greatest lower bound of S (i.e. $a = \inf S$) if and only if a is a lower bound of S and for every $\epsilon > 0$ there exists $x \in S$ such that $a + \epsilon > x$. (6)
- 2.3 Let x be a real number. Prove that there exists an integer n such that $n \leq x < n + 1$. (4)
- 2.4 Prove the theorem of Eudoxus: Between any two real numbers there is a rational number. (5)
- [21]

Question Three

- 3.1 Let $\epsilon > 0$. Prove that $|a| > \epsilon$ if and only if $a > \epsilon$ or $a < -\epsilon$. (3)
- 3.2 Prove that $||a| - |b|| \leq |a - b|$ for any real numbers a and b . (3)
- 3.3 (a) Define what is meant by "a subset A of \mathbb{R} is open in \mathbb{R} ". (1)
 (b) Let $a, b \in \mathbb{R}$ with $b > a$. Prove that the interval (a, b) is an open subset of \mathbb{R} . (4)
- 3.4 (a) Prove that the intersection of a finite number of open sets in \mathbb{R} is open, (3)
 (b) Prove that the union of any number of open sets in \mathbb{R} is open. (3)
- 3.5 Show from definition that the interval $[a, b]$ is closed in \mathbb{R} . (2)
- 3.6 (a) Define what is meant by " x is an interior point of A ". (1)
 (b) Let $A \subseteq \mathbb{R}$. Prove that A° is an open subset of \mathbb{R} . (4)
- [24]

Question Four

- 4.1 (a) Let $S \subseteq \mathbb{R}$. Define an accumulation point of S . (1)
 (b) Let $S \subseteq \mathbb{R}$. Prove that S is closed if and only if S contains all its accumulation points. (5)

- 4.2 **State and prove** the Nested Interval Property. (5)
- 4.3 **State, without proof,** the Bolzano-Weierstrass Theorem. (2)
- 4.4 (a) Let $\{a_n\}$ be a sequence of real numbers. Define what is meant by “the sequence converges to a real number a ” using ϵ . (2)
- (b) (i) Prove that if the sequence $\{a_n\}$ converges to a , then it is bounded. (4)
- (ii) Give a contrapositive of the above statement in (b) (i). (4)
- 4.5 Use ϵ and δ to show that $\lim_{x \rightarrow 0} \sin(1/x)$ does not exist. (4)
- [23]

Question Five

- 5.1 (a) Define a Cauchy sequence. (1)
- (b) Prove that every Cauchy sequence of real numbers is convergent. (5)
- 5.2 Prove that $\overline{Q^c} = \mathbb{R}$ where Q^c is the set of all irrational numbers. (3)
- 5.3 (a) Let $f : E \rightarrow \mathbb{R}$, where $E \subseteq \mathbb{R}$. Use ϵ and δ to define what is meant by f is continuous at x_0 in E . (2)
- (b) Let f be continuous at x_0 . Prove that f is bounded in some neighbourhood of x_0 . (3)
- (c) Let $f, g : D \rightarrow \mathbb{R}$ be continuous functions at x_0 . Prove that (i) $f + g$ and (ii) fg are both continuous at x_0 . (8)
- [22]

Question Six

- 6.1 Let $f : [-5, 3] \rightarrow \mathbb{R}$ be given by $f(x) = \frac{3}{x-2}$. Prove that f is uniformly continuous on $[-5, 3]$, **from definition**. (4)
- 6.2 Let D be a compact subset of \mathbb{R} and $f : D \rightarrow \mathbb{R}$ be a continuous function. Prove that f is uniformly continuous. (5)

- 6.3 (a) State, without proof, the Mean-value Theorem. (2)
(b) Let $f : [a, b] \rightarrow \mathbf{R}$ be continuous and differentiable on (a, b) . Prove the following statements:
(i) If $f'(x) \neq 0$ for all $x \in (a, b)$, then f is one-to-one.
(ii) If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.
(iii) If $f'(x) < 0$ for all $x \in (a, b)$, then $x < y$ implies $f(x) > f(y)$ for x and y in (a, b) . (6)
- 6.4 Let $F = \{\{1, 3\}, \{3\}, \{3, 4\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}\}$. Partially order F with set inclusion \subseteq . (a) Say, with reasons, whether F is totally ordered or not; (b) List all the minimal and maximal elements of F , if any. (4)
- 6.5 (a) Write down the Maclaurin series of $f(x) = \frac{1}{1-x}$ and determine its region of convergence.
(b) Deduce from (a) above the Maclaurin series for $\arctan x$, stating its region of convergence. Hence use the first 4 terms of the series to approximate $\arctan \frac{3}{4}$ to 4 decimal places. (5)

[26]

END