University of Fort Hare

MAT 224

Degree Examinations: November 2018

Subject: Mathematics 2
Paper: Real Analysis

Time: 3 Hours Marks: 100 Subminimum: 40

This question paper consists of 4 pages

Internal examiner(s)

Prof B B Makamba
Dr S Ngcibi

External examiner(s)

Instructions

Answer NO more than FIVE (5) questions. Symbols used have the usual meanings.

*

Mathematics Exams November 2018

Course Code: MAT 224 **Course Name:** Real Analysis
Time: 3 Hours **Marks:** 100 **Subminimum: 40**

Instructions: Answer **FIVE** questions. **This paper consists of 4 pages**

Question One

1.1 Let A; B and C be sets. Prove that (a) $A-(B\cup C)=(A-B)\cap(A-C)$; (b) $A\subseteq B$ if and only if $A\cap B^c=\emptyset$, where $A^c=X-A$. (6)

1.2 If A, $B\subseteq X$, we write B^c for $X-B$ and A^c for $X-A$. Prove that $A\subseteq B$ iff $B\cup A^c=X$. (3)

1.3 (a) Define what is meant by "set A is equivalent (equipotent) to set B". (1)

(b) Show (from definition) that the interval $(0,1)$ is equivalent to the set R of all real numbers. (4)

1.4 (a) Define what is meant by a "countably infinite set" and hence define a countable set. (2)
(b) Prove that the set Q of all rational numbers is countable. (4)

1.5 (a) Let S be a bounded set. Define what is meant by (i) the supremum of S and (ii) the infimum of S. (3)
(b) Let $S=\{x:\ 3x^2-5x\leq 2\}$. Find (i) sup S and inf S, if they exist. (3)

[26]

Question Two

2.1 Prove the following: Let $S\subseteq\mathbb{R}$. A real number a is the least upper bound of S (i.e. $a=\sup S$) if and only if a is an upper bound of S and for every $\epsilon>0$ there exists $x\in S$ such that $a-\epsilon<x$. (6)

2.2 Let x be a real number. Prove that there exists an integer n such that $n \leq x < n+1$. (4)

2.3 Prove the theorem of Eudoxus: Between any two real numbers there is a rational number. (5)

2.4 (a) Let $\epsilon > 0$. Prove that $|a| < \epsilon$ if and only if $-\epsilon < a < \epsilon$. (3)
(b) Let x, y, a, $b \in \mathbb{R}^+$ and suppose $\frac{x}{y} < \frac{a}{b}$. Prove that $\frac{x}{y} < \frac{x+a}{y+b} < \frac{a}{b}$. (3)

2.5 Prove that $||a| - |b|| \leq |a-b|$ for any real numbers a and b. (3)

[24]

Question Three

3.1 (a) Define what is meant by "a subset A of $\mathbb{R}$ is open in $\mathbb{R}$". (1)
(b) Let $a \in \mathbb{R}$. Prove that the interval $(-\infty, a)$ is an open subset of $\mathbb{R}$. (3)
(c) Let $a, b \in \mathbb{R}$ with $a < b$. Show that the interval $(a, b]$ is NOT an open subset of $\mathbb{R}$. (2)

3.2 (a) Prove that the union of any number of open sets in $\mathbb{R}$ is open. (3)
(b) Prove that the intersection of a finite number of open sets in $\mathbb{R}$ is open, (3)

3.3 Show from definition that the interval $[a, \infty)$ is closed in $\mathbb{R}$. (2)

3.4 (a) Define what is meant by "x is an interior point of A". (1)
(b) Let $A \subseteq \mathbb{R}$. Prove that A^o is an open subset of $\mathbb{R}$. (4)

3.5 (a) Let $S \subseteq \mathbb{R}$. Define an accumulation point of S. (1)
(b) Let $S \subseteq \mathbb{R}$. Prove that S is closed if and only if S contains all its accumulation points. (5)

[25]

Question Four

4.1 **State, without proof,** the Nested Interval Property. (2)

4.2 State and **prove**, the Bolzano-Weierstrass Theorem. (6)

4.3 (a) Let $\{a_n\}$ be a sequence of real numbers. Define what is meant by "the sequence converges to a real number a" using ϵ . (2)
(b) (i) Prove that if the sequence $\{a_n\}$ converges to a, then it is bounded.
(ii) Give a contrapositive of the above statement in (b) (i). (4)
(c) Show that the seq $\{a_n : n = 1, 2, \cdots\}$ is unbounded, where $a_1 = 1$; $a_2 = 1 + 1/2$; $a_3 = 1 + 1/2 + 1/3$; $\cdots$, $a_n = 1 + 1/2 + 1/3 + \cdots + 1/n$. (3)

4.4 (a) Define a Cauchy sequence. (1)
(b) Prove that every Cauchy sequence of real numbers is convergent. (5)

[23]

Question Five

5.1 Prove that $\overline{Q} = \mathbb{R}$ where Q is the set of all rational numbers. (3)

5.2 (a) Let $f : E \to R$, where $E \subseteq R$. Use ϵ and δ to define what is meant by f is continuous at x_o in E. (2)
(b) Let f be continuous at x_o. Prove that f is bounded in some neighbourhood of x_o. (3)
(c) Let f, $g : D \to \boldsymbol{R}$ be continuous functions at x_0. Prove that (i) $f + g$ and (ii) fg are both continuous at x_0. (8)

5.3 Let $f : [-5, 3] \to \boldsymbol{R}$ be given by $f(x) = \frac{3}{x-2}$. Prove that f is uniformly continuous on $[-5, 3]$, **from definition**. (4)

5.4 Let D be a compact subset of R and $f : D \to \boldsymbol{R}$ be a continuous function. Prove that f is uniformly continuous. (5)

[25]

Question Six

6.1 Show that the equation $x = \cos x$ has at least one solution in $[0, \frac{\pi}{2}]$ (3)

6.2 (a) State, withou proof, the Mean-value Theorem. (2)
(b) Let $f : [a, b] \to \boldsymbol{R}$ be continuous and differentiable on (a, b). Prove

the following statements:
(i) If $f'(x) \neq 0$ for all $x \in (a,b)$, then f is one-to-one.
(ii) If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant.
(iii) If $f'(x) < 0$ for all $x \in (a,b)$, then $x < y$ implies $f(x) > f(y)$ for x and y in (a,b). (6)

6.3 Give geometric interpretations of statements (i), (ii), (iii) in Question 6.2 (b) above. (5)

6.4 Let $F = \{\{1, 3\}, \{3\}, \{3, 4\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}\}$. Partially order F with set inclusion $\subseteq$. (a) Say, with reasons, whether F is totally ordered or not; (b) List all the minimal and maximal elements of F, if any. (4)

6.5 (a) Write down the Maclaurin series of $f(x) = \frac{1}{1-x}$ and determine its region of convergence.
(b) Deduce from (a) above the Maclaurin series for $\arctan x$, stating its region of convergence. Hence use the first 4 terms of the series to approximate $\arctan \frac{2}{3}$ to 4 decimal places. (5)

[25]

END