UNIVERSITY OF FORT HARE

MATHEMATICAL MODELLING

MAP 513

JUNE EXAMINATIONS

Time: 3 hours

Subject: Applied Mathematics

Marks: 180 (182 available)

This paper consists of 6 pages

including the cover page.

Examiner: **Dr. S. J. Childs**

External examiner: **Prof. K. Dukuza**

INSTRUCTIONS

All questions may be answered.

Show your working.

Sloppy work will be penalised.

1. Kermack and McKendrick's famous SIR equations of 1927 can be stated as

$$\begin{aligned}\frac{dS}{dt} &= -\beta SI \\ \frac{dI}{dt} &= \beta SI - \gamma I \\ \frac{dR}{dt} &= \gamma I,\end{aligned}$$

in which S denotes the fraction of a population, N, still susceptible to infection, β denotes the rate of infection, I denotes the fraction already infected, R denotes the 'recovered' fraction that have either acquired immunity, or died from the disease, and γ denotes this rate of recovery, or death.

(a) Write a very brief exposition of the quantity r_0 and explain its significance when a naive population is infected. (12 marks)

(b) State and prove the threshold theorem. (12 marks)

(c) For a naive population ($R = 0$) in the abscence of an epidemic, at least how many potential hosts, need to be vaccinated? (4 marks)

(d) Derive an expression for the minimum fraction of the population that needs to be vaccinated, V, in terms of the total fraction infected to date, T, once an epidemic has broken out. (21 marks)

(e) Sketch an explanatory graph, illustrating all relevant quantities referred to in your answer to (1d), above. (10 marks)

2. The Lotka-Volterra, predator-prey model can be stated as

$$\frac{dx}{dt} = \alpha x - \beta xy$$
$$\frac{dy}{dt} = \delta xy - \gamma y,$$

in which t is time, x denotes the population density of a prey species, with growth rate α and predation rate β, while y denotes the population density of the predator, with a prey-dependent growth rate δ and mortality rate γ.

(a) Work out the equilibrium points. (11 marks)

(b) Comment on the stability of these equilibrium points. (2 marks)

3. (a) A particle path is the path traced out by a given paticle. What are streamlines? (4 marks)

(b) Define a steady flow in terms of the answer to (3a), above. (1 mark)

(c) The conservation of linear momentum, for a fluid, can be stated as

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\nabla\mathbf{v}\ \mathbf{v} = \rho\mathbf{b} + \text{div}\boldsymbol{\sigma},$$

in which $\mathbf{v}$ is the velocity, $\mathbf{b}$ is the body force per unit mass and $\boldsymbol{\sigma}$ is the stress. Derive the famous Bernoulli's equation. (17 marks)

(d) A solid sphere of radius R moves through an infinite expanse of fluid at a velocity, $\mathbf{u}$. Determine the velocity field of the fluid. (43 marks)

4. Consider a rigid body in motion, in terms of some, general reference, $\hat{R}(\hat{o}, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$. Let $\hat{\mathbf{x}}$ denote the position vector of its centre of mass and $\hat{\mathbf{x}} + \hat{\mathbf{r}}$ the position vector of some, other, material point within it. Then the principle of conservation of angular momentum can be stated as

$$\frac{d\hat{\mathbf{q}}}{dt} = \Sigma(\hat{\mathbf{r}} \wedge \hat{\mathbf{b}}), \qquad (1)$$

in which $\hat{\mathbf{q}}$ is the angular momentum and $\Sigma(\hat{\mathbf{r}} \wedge \hat{\mathbf{b}})$ is the sum of the torques which arise as a result of the external forces acting about the centre of mass ($\hat{\mathbf{b}}$ is the force acting at $\hat{\mathbf{x}} + \hat{\mathbf{r}}$).

(a) Reformulate

$$\hat{\mathbf{q}} = \int_{\hat{\Omega}} \rho\, \hat{\mathbf{r}} \wedge \hat{\mathbf{v}}\, d\hat{\Omega},$$

in terms of a new reference, $R(o, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, using the famous Coriolis theorem. (4 marks)

(b) Discuss two attributes this new reference could be chosen to have which would greatly simplify the new expression you obtained for q. (2 marks)

(c) Show that

$$\mathbf{q} = \mathbf{J}\boldsymbol{\omega}$$

in the new reference, where ω is the angular velocity and $\mathbf{J}$ is the inertia matrix. (18 marks)

(d) What would the axes have to coincide with, in order for the aforementioned relationship to be simplified to

$$\mathbf{q} = \begin{bmatrix} J_{11}\omega_1 \\ J_{22}\omega_2 \\ J_{33}\omega_3 \end{bmatrix}?$$

(2 marks)

(e) Show that the conservation of angular momentum, equation (1), may be expressed as

$$\begin{aligned} J_{11}\frac{d\omega_1}{dt} + (J_{33} - J_{22})\omega_2\omega_3 &= \Sigma(\mathbf{r} \wedge \mathbf{b}) \cdot \mathbf{e}_1 \\ J_{22}\frac{d\omega_2}{dt} + (J_{11} - J_{33})\omega_3\omega_1 &= \Sigma(\mathbf{r} \wedge \mathbf{b}) \cdot \mathbf{e}_2 \\ J_{33}\frac{d\omega_3}{dt} + (J_{22} - J_{11})\omega_1\omega_2 &= \Sigma(\mathbf{r} \wedge \mathbf{b}) \cdot \mathbf{e}_3 \end{aligned}$$

in the new, more appropriate reference. (4 marks)

5. (a) A "cold front" is a weather phenomenon in which a cold, low-pressure cell is formed near the poles, then moved to lower latitudes. The adjacent warm air tends to rise, while the cold, frontal air flows in, underneath it, replacing it. Ignoring Coriolis affects, in what direction will the flow of this cold air be, relative to the centre of the low pressure system? (1 mark)

(b) In what direction will water flow, relative to an open plug, at the equator? (1 mark)

(c) How do these two directions compare? (1 mark)

(d) Given that our planet spins from west to east, use the famous Coriolis theorem

$$\frac{\partial \hat{\mathbf{v}}}{\partial t} = \frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\omega} \wedge \mathbf{v},$$

(in which $\boldsymbol{\omega}$ is the angular velocity B relative to $\hat{B}$) and the right-hand rule to determine whether the spin observed from above should be clockwise or anti-clockwise in the case of:

i. Water flowing out a plug in the Southern Hemisphere. (2 marks)

ii. A cold front the Southern Hemisphere. (2 marks)

iii. Water flowing out a plug in the United States. (2 marks)

iv. A cold front in Europe. (2 marks)

(e) If one considers the purely physical interaction between the boundary layer, or surroundings, in what sense should the flow tend to rotate for:

i. A divergent flow in the Southern Hemisphere? (1 mark)

ii. A convergent flow in the Southern Hemisphere? (1 mark)

iii. A divergent flow in the Northern Hemisphere? (1 mark)

iv. A convergent flow in the Northern Hemisphere? (1 mark)