

UNIVERSITY OF FORT HARE

Partial Differential Equations
MAP 321

Degree Examinations

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This question paper consists of 3 pages

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Instructions

Answer **any 4** questions.
Symbols have the usual meanings

Question 1

1.1 Consider the following general partial differential equation

$$a \frac{\partial u}{\partial t} + b \frac{\partial u}{\partial x} = f(x, t) \text{ where } \begin{matrix} x \in \mathbb{R}, t \geq 0 \\ u(x, 0) = g(x), x \in \mathbb{R} \end{matrix}$$

and

(i) show that $u(x, t) = g\left(x - \frac{b}{a}t\right) + \frac{1}{a} \int_0^t f\left(x - \frac{b}{a}t + \frac{b}{a}\tau, \tau\right) d\tau$ is the solution, hence

(ii) solve $4 \frac{\partial u}{\partial t} + 2 \frac{\partial u}{\partial x} = \cos t$ where $u(x, 0) = \sin x$, (8, 5)

1.2 Find the general solution to the following differential equation

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 2 \sin(3x + 2y). \tag{12}$$

[25]

Question 2

2.1 Describe the regions in the x-y plane where the following differential equation is elliptic, parabolic or

hyperbolic; $(x^2 - 1) \frac{\partial^2 U}{\partial x^2} + 2xy \frac{\partial^2 U}{\partial x \partial y} + (y^2 - 1) \frac{\partial^2 U}{\partial y^2} = x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y}$. (4)

2.2 Use Laplace transforms to solve the following PDE,

$$\frac{\partial U}{\partial t} + \frac{\partial^2 U}{\partial x^2} = 0,,$$

$$U(0, t) = U(4, t) = 0,,$$

subject to $U(x, 0) = 6 \sin \frac{\pi}{2} x + 3 \sin \pi x$ (9)

2.3 Use the method of separation of variables to solve the following boundary value problem,

PDE $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \quad 0 < x < \pi, \quad 0 < y < b,$

subject to $U(0, y) = U(\pi, y) = 0, \quad 0 < y < b,$

$$U(x, b) = 0, \quad U(x, 0) = 3 \sin x, \quad 0 < x < \pi$$

NB: It has a particular solution of the form, $Z(x, y) = \frac{3 \sin x \sinh(b - y)}{\sinh b}$. (12)

[25]

Question 3

3.1 Prove that a necessary condition for

$$w = f(z) = u(x, y) + iv(x, y)$$

to be analytic in a region is that the Cauchy equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{be satisfied in the region.} \quad (4)$$

3.2 If $V(x, y) = 2x(1 - y)$, find $U(x, y)$ the harmonic conjugate such that $f(z) = U(x, y) + V(x, y)$ is an analytic function of z . (4)

3.3 Prove that, $\oint_C \frac{dz}{(z-a)^n} = \begin{cases} 2\pi i & \text{if } n=1 \\ 0 & \text{if } n=2, 3, 4, \dots \end{cases}$ where C is a simple closed curve bonding a region having $z = a$ as interior point. (5)

3.4 Evaluate the integral $\int_{|z|=\frac{3}{2}} \frac{z^3 + 1}{(z+1)^2(z^2+1)} dz$ using the Cauchy's integrals formulas. (5)

3.5 Show that $\int_0^{\infty} \frac{\cos mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}$ for $m > 0$. (7)

[25]**Question 4**

4.1 Define a conformal mapping $w = f(z)$ at a point z . (2)

4.2 Consider the following complex function, $f(z) = \frac{1}{z}$, and

4.2.1 And find the image of the disk $|z| < 1$, (show the diagrams) (3)

4.2.2 Find the image of the region $\operatorname{Re} z > \frac{1}{2}$ (show the diagrams). (3)

4.2.3 Find the image of the upper half plane $f(z)$. (show the diagrams). (3)

4.3 Given $f(z) = z^2$, find the image of the square $S = \{x + iy \mid 0 < x, y < 3\}$, (6)

4.4 Use the transformation, $T(z) = i \frac{(1-z)}{(1+z)}$ to find the image of the disk $|z| < 1$, (4)

4.5 A function satisfying the Laplace's equations is said to be conformal and harmonic, show that the function $x^2 - y^2 + 2y$ is harmonic in the w plane under the transformation $z = w^3$. (4)

[25]

Question 5

5.1 Explain the fundamental problem of the Calculus of variations, and hence derive the Euler's equations. (12)

5.2 Find (a) the extremals of $\int_{x_1}^{x_2} \sqrt{1+(y')^2} dx$ and thus (b) show that the shortest distance between the two points in a plane is a straight line. (13)
[25]
