UNIVERSITY OF FORT HARE

MAT 507

HONOURS EXAMINATIIONS

January 2019

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Time: 3 HOURS

Subject: Functional Analysis

Marks: 100

This paper consists of 3 pages including cover page

Internal Examiner

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Instructions

All questions may be answered

Symbols used, unless stated otherwise, have their usual meaning

Sloppy work will be penalized

Question 1

1. Let S be a subspace of a linear space X.

   (a) (i) Define precisely what it means for S to be **absolutely convex**; (ii) What is meant by co(S), the **convex hull** of S? **[3,2]**

   (b) Show that if S is a nonempty subset of a linear space X, then co(S) is the set of all convex combinations of elements of S. That is, [8]

$$\text{co}(S) = \left\{ \sum_{j=1}^{n} \lambda_j x_j : x_1, x_2, \ldots, x_n \in S, \lambda_j \geq 0, \sum_{j=1}^{n} \lambda_j = 1, n \in \mathbb{N} \right\}.$$

2. (a) Let p and q be positive real numbers. Define what it means for p and q to be *conjugate exponents.* [3]

   (b) For ℓ_p denoting the set of all sequences $(x_n)_1^\infty$ of real or complex numbers satisfying the condition $\sum_{i=1}^{\infty} |x_i|^p < \infty$, consider $(x_n) \in \ell_p$ and $(y_n) \in \ell_q$, where $p > 1$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Show, quoting accurately any result(s)you use, that

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} |y_i|^q \right)^{\frac{1}{q}}$$

[7]

   (c) State the *Minkowski's Inequality* for sequences. [3]

3. (a) Show that all norms on a finite-dimensional normed linear space are equivalent.

   (b) Use (a) above to show that any linear operator on a finite-dimensional normed linear space is continuous. **[6,4]**

Question 2

1. (a) Define what is meant by a *Banach Space.* [3]

   (b) Prove that if every *absolutely convergent* series in a normed linear space $(X, ||\cdot||)$ is convergent, then X is a *Banach space.* [7]

2. Prove that every finite-dimensional normed linear space X is *complete.* [7]

3. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space over some field $\mathbb{F}$.

   (a) Define what is meant by a *Hilbert Space.* [3]

   (b) Show that if M is a linear subspace of $\mathcal{H}$, then $\overline{M} = M^{\perp\perp}$. [5]

   (c) Deduce from (a) that a linear subspace M of $\mathcal{H}$ is closed if and only if $M = M^{\perp\perp}$. [3]

4. Define an operator $T : \ell_2 \longrightarrow \ell_2$ by

$$Tx = T(x_1, x_2, x_3, \ldots) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right), \text{ where } x = (x_n)_{n=1}^{\infty} \in \ell_2.$$

Show that T is a bounded linear operator and find its norm. [7]

Question 3

1. Let X and Y be normed linear spaces over a field $\mathbb{F}$ and let $T \in \mathcal{B}(X, Y)$. Prove that [7]

$$||T|| = \sup\left\{\frac{||Tx||}{||x||} : x \neq 0\right\} = ||T|| = \sup\{||Tx|| : ||x|| = 1\} = \sup\{||Tx|| : ||x|| \leq 1\}.$$

2. (a) State and prove the Riesz-Fréchet Representation Theorem on bounded linear functionals on a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. [7]

   (b) Let $f : \ell_2 \to \mathbb{C}$ be defined by

$$f(x) = x_1 + \frac{x_3}{3^2} + \frac{x_5}{5^2} + \frac{x_7}{7^2} + \cdots, \text{ where } x = (x_n)_{n=1}^{\infty} \in \ell_2.$$

   Show that f is a bounded linear functional on ℓ_2. [5]

3. Show that the dual of ℓ_1 is isometrically isomorphic to ℓ_∞; that is, $\ell_1^* \cong \ell_\infty$. [7]

4. (a) State the *Hahn-Banach Theorem* for normed linear spaces. [2]

   (b) What is meant by a *reflexive* normed linear space. [3]

   (c) Prove that a closed linear subspace of a reflexive space is reflexive. [5]

END OF EXAMINATION