

MAT 122

SUPPLEMENTARY EXAMINATIONS

December 2024

Subject: **Mathematics 1**

Paper: A Practical Approach to Differential Calculus

Time: **3 Hours**

Marks: **100**

This question paper consists of 3 pages

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Instructions

Answer all questions.
Symbols have the usual meanings

Question One [25 marks]

1.1 Prove by Mathematical Induction that,

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n \quad \text{for } n \in \mathbf{Z}^+ \quad (4)$$

1.2 Solve the inequality and represent your solution on a number line $x^2 < 2x + 8$ (4)

1.3 Find $f \circ g$ if $f(x) = \sqrt{x-3}$, $g(x) = x^2$, (4)

1.4 (a) State and prove the Binomial Theorem using Mathematical Induction. (6)

(b) If one of the terms of the binomial expansion $\left(\frac{1}{x^2} - x\right)^{100}$ is βx^{-80} , find β and express it in the form $\frac{n!}{(n-r)!r!}$. (4)

1.5 Solve the Absolute value inequality $|x - 5| > 2$ (3)

Question Two [25 marks]

2.1 Given a function $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ cx^2 + d & \text{if } 1 \leq x \leq 2 \\ 4x & \text{if } x > 2 \end{cases}$

Find c and d so that f is continuous (6)

2.2 State the Squeeze Theorem. Use Squeeze Theorem to prove $\lim_{x \rightarrow 0} x^3 \cos\left(\frac{1}{x}\right) = 0$ (7)

2.3 Determine whether f is even, odd, or neither for the following:

(a) $f(x) = x^5 + x$

(b) $f(x) = 1 - x^4$

(c) $f(x) = 2x - x^2$ (6)

2.4 Sketch the graph of f , write its **DOMAIN** and **RANGE**. If

$$f(x) = \sqrt{2x-1} \quad (6)$$

Question Three [25 marks]

3.1 From definition, find $f'(x)$ for the function $f(x) = \sin x$ (3)

3.2 Use implicit differentiation to find $\frac{dy}{dx}$ if $x^4 y^2 - x^3 y + 2xy^3 = 0$ (4)

3.3 Find $\frac{dy}{dx}$ if (a) $y = \sin(\cos x)$ (3)

(b) $y = e^{x^2} + \ln(2x^3 + 5)$ (4)

(c) $y = \frac{x-1}{x+1}$ (4)

3.4 Evaluate the following limits.

$$(a) \lim_{x \rightarrow 0} \frac{x-5}{x^2-25} \quad (3)$$

$$(b) \lim_{x \rightarrow 0} \frac{\sin 8x}{\sin 9x} \quad (4)$$

Question Four [25 marks]

4.1 Find the maximum and minimum values of $y = x^3 - 12x + 2$, distinguishing between them. Sketch the graph of the function. (4)

4.2 If $xy = 8$, find the minimum value of $V = x + 2y$, where x and y are positive (4)

4.3 If $P = \sqrt[3]{6t^2 - 1}$ and t is increased by a small amount. Find an approximation for the corresponding increase in P . Calculate the approximate **percent** increase in P if there is a 3% increase in t when $t = 4$. (4)

4.4 Find all functions g such that $g'(x) = 4 \sin x + \frac{2x^5 - \sqrt{x}}{x}$ (4)

4.5 Differentiate $y = \left(\frac{1 - \cos 2x}{1 + \cos 2x} \right)^4$ (5)

4.6 Find the dimensions of the rectangle with a perimeter of 200ft with an area as large as possible. (4)

