MAT 311

DEGREE EXAMINATIONS

June 2023

Subject: **Mathematics 3**
Paper: Abstract Algebra

Time: **3 Hours** Marks: **100**

This question paper consists of 5 pages including the cover page

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Instructions

Answer FIVE (5) questions.
Symbols used have the usual meanings

Question One

1.1 Let a, b, c be positive integers such that a and c are relatively prime. If a divides bc, show that a divides b. (3)

1.2 (a) Define a group. (2)
(b) Discuss an example of a group of order 4 having the property that each of its non-identity elements has order 2. Further, argue that such a group cannot be cyclic. Draw a lattice diagram for the subgroups of such a group. (5)

1.3 Fix a positive integer m and define a binary operation $\star$ on Q^+ (the set of all positive rational numbers) by $x \star y = m(xy)$ for all positive rational number x and y. Show that $(Q^+, \star)$ is a group. (3)

1.4 Show that a group that has only a finite number of subgroups must be a finite group. (5)

1.5 Let $S = R - \{1\}$. Define $\star$ on S by $a \star b = a + b - ab$. (a) Show that $\star$ gives a binary operation on S; (b) Show that S with $\star$ is a group; (c) Find the solution of the equation $5 \star x \star 4 = 25$ in S. (6)

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Question Two

2.1 Discuss and find all the subgroups of the group Z_{12} and then draw a lattice diagram for the subgroups . (4)

2.2 Prove that a subgroup of a cyclic group is cyclic. (5)

2.3 Give an example, with explanation, of a non-abelian group whose proper subgroups are all cyclic. (Ensure that you show that your group is non-abelian and that indeed all its proper subgroups are cyclic). (5)

2.4 Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 2 & 1 \end{pmatrix}$ in S_5. Find (a) α^{-1} ; (b) the order of α;

(c) α^{101} . (6)

2.5 (a) Express $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 6 & 1 & 7 & 2 & 3 & 5 \end{pmatrix}$ in S_7 as (i) a product of disjoint cycles; (ii) a product of transpositions.
(b) Is α even or odd? Explain. (4)

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Question Three

3.1 (a) Show that the group $Z_p \times Z_p$ is not cyclic for any prime p. (3)
(b) Compute the order of 8 in the group Z_{12} under addition modulo 12. (2)

3.2 State and prove Lagrange's Theorem. Deduce that every group of prime order p is cyclic. (6)

3.3 Let G be a group of order 91. Argue that every proper subgroup of G is cyclic. (3)

3.4 Find the number of left cosets of $H =< (3, 6) >$ in the group $Z_{30} \times Z_{24}$, without enumerating the elements of H. (3)

3.5 (a) Let G be a group. Define a normal subgroup of G.
(b) Let $f : G \rightarrow H$ be a homomorphism from a group G into a group H. Define the kernel of f, and then prove that the kernel of f is a normal subgroup of G. (5)

3.6 Show that if a finite group G has exactly one subgroup H of a given order n, then H is normal in G. (2)

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Question Four

4.1 (a) Compute the order of the coset $(2, 4) + < (5, 2) >$ in the group $(Z_{10} \times Z_8)/ < (5, 2) >$. (3)

(b) Compute the group $(Z_{10} \times Z_{12})/ < (2,3) >$ (i.e. find the simplest group to which it is isomorphic). (4)

4.2 Show that the group $(Z, +)$ is isomorphic to the group $(3Z, +)$. (3)

4.3 (a) Define the center $Z(G)$ of a group G. (1)
(b) Show that (i) $Z(G)$ is a subgroup of G and (ii) $Z(G)$ is normal in G. (5)

4.4 Let G be a group and $Z(G)$ be the center of G. If $G/Z(G)$ is cyclic, prove that G is abelian. (4)

4.5 (a) Define the commutator subgroup G' of a group G. (1)
(b) Show that the commutator subgroup G' of G is normal in G. (3)

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Question Five

5.1 (a) Define an ideal of a ring R. (2)
(b) Define a prime ideal of a commutative ring R. (2)
(c) Define a maximal ideal of a commutative ring R. (2)

5.2 (a) Let R be a commutative ring with unity. Show that every maximal ideal M of R is also a prime ideal. (2)
(b) Find a non-trivial proper ideal of $Z \times Z$ that is not prime, giving reasons. (2)
(c) Discuss an example showing that a factor ring of a non-integral domain may be an integral domain. (3)

5.3 Let R be a ring with unity and N an ideal of R containing a unit. Prove that $N = R$. (3)

5.4 Let R be a commutative ring with unity. Prove that the set of all units in R is a group under the ring multiplication. (4)

5.5 Find the characteristic of the ring $Z_8 \times Z_6$. (2)

5.6 Show directly that the ring Z_5 is a field. (2)

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Question Six

6.1 Let R be the ring $Z \times Z$ and $N = 2Z \times Z$. Show that N is a prime ideal of R. (3)

6.2 Let F be the ring of all real-valued functions on R, and let N be the subring of F consisting of all the constant functions of F. Show that N is not an ideal of F. (2)

6.3 Prove that if F is a field, then every ideal in F[x] is principal. (6)

6.4 Find a non-integral domain whose quotient is a field. (2)

6.5 Is the subring $Z \times 9Z$ a prime ideal of $Z \times Z$? Why?. (3)

6.6 Show that each homomorphism of a field is either one-to-one or maps everything onto 0, where 0 is the additive identity element. (4)

6.7 Is $Q[x]/ < x^2 - 6x + 6 >$ a field? Explain. (Q is the field of all rational numbers). (2)

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