UNIVERSITY OF FORT HARE

Complex Analysis
MAT 322

Examination

November

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Time: 3 hours

Subject: Complex Analysis

Maximum Marks: 100

This question paper consists of 4 pages

Internal examiner

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Instructions
Answer **Five Questions**

Symbols have the usual meanings.

Nov 2024: Module Code: MAT 322

QUESTION 1 [28]

Choose at least **one correct answer** and **write** your **answer on** the **question paper** and **hand it in with** your **examination book.**

1.1) The **image** of the half-plane $\boldsymbol{Re(z) > 1}$ under the linear transformation $\boldsymbol{w = f(z) = (1-i)z + 1 - 2i}$ is given by

(A) $v > u - 5$ **(B)** $v > u + 5$ **(C)** $u > v + 5$ **(D)** $u > v + 1$ **(E)** None of the above.

1.2) $\lim\limits_{z \to 1=i} \frac{z^2+z-1-3i}{z^2-2z+2}$

(A) $-1 + \frac{3}{2}i$ **(B)** $1 + \frac{3}{2}i$ **(C)** $1 - \frac{3}{2}i$ **(D)** $-1 - \frac{3}{2}i$ **(E)** None of the above.

1.3) The function $f(z) = x^2 + y^2 + i2xy$ is differentiable on ...

(A) $\{x + iy | x \in \mathbb{R}, y = 0\}$ **(B)** $\{x + iy | x = 1, y = 0\}$ **(C)** complex plane **(D)** nowhere
(E) None of the above.

1.4) $\left\{\frac{(n+i)(1-ni)}{n^2}\right\}$

(A) Is divergent. **(B)** converges to $1 - i$. **(C)** converges to 0. **(D)** converges to i.
(E) None of the above.

1.5) $\sum_{n=1}^{\infty}\left(\frac{1+i}{3}\right)^n =$

(A) $\frac{2+i}{5}$ **(B)** $\frac{6+3i}{5}$ **(C)** $\frac{6-3i}{5}i$ **(D)** $\frac{-6-3i}{5}i$ **(E)** None of the above.

1.6) The disk of convergence for the series $\sum_{n=0}^{\infty}\left(\frac{z+i}{6+8i}\right)^n$ is

(A) $|z + i| < 10$ **(B)** $|z + i| < 5$ **(C)** $|6 + 8i| < 5$ **(D)** $|z + i| < 1$ **(E)** None of the above.

1.7) Find all values of $z \in \mathbb{C}$ for which $e^z = -1 - i$ hold.

(A) $\ln(\sqrt{2}) + i\left(\frac{\pi}{4}\right)$ **(B)** $\ln(\sqrt{2}) + i\left(\frac{\pi}{4} + 2\pi n\right), n \in \mathbb{Z}$ **(C)** $\ln(\sqrt{2}) + i\left(\frac{3\pi}{4} + 2\pi n\right), n \in \mathbb{Z}$ **(D)** none
(E) None of the above.

1.8) Find the principal value of $\boldsymbol{i^i}$.

(A) $e^{-i\left(\frac{\pi}{2}+2\pi n\right)}, n \in \mathbb{Z}$ **(B)** $e^{-i\left(\frac{\pi}{2}+2\pi n\right)}, n \in \mathbb{Z}$ **(C)** $e^{-i\frac{\pi}{2}}$ **(D)** $e^{-\frac{\pi}{2}}$ **(E)** None of the above.

1.9) $\int_0^2 \frac{t}{t+i} dt=$

(A) $2 - \arctan(2) - i\ln(\sqrt{5})$ **(B)** $2 - i\ln\sqrt{5}$ **(C)** $2 + i\ln\sqrt{5}$ **(D)** $\ln(t+i)$
(E) None of the above.

1.10) Evaluate $\int_C z\, dz$ where C is the line segment from i to 1 given by the parametrization $z(t) = 1 - t + it, 0 \leq t \leq 1$.

(A) i **(B)** $-i$ **(C)** -1 **(D)** 1 **(E)** None of the above.

[10x3=30]

QUESTION 2

2.1) Let $f(z) = \frac{z^2}{|z|^2} = \frac{x^2-y^2-2ixy}{x^2+y^2}$.

2.1.1) Compute $\lim_{z\to 0} f(z)$ along the line $y = 2x$.
2.1.2) Compute $\lim_{z\to 0} f(z)$ along $y = x^2$.
2.1.3) Use your answers above to explain whether $\lim_{z\to 0} f(z)$ **exists**? **[3, 3, 1]**

2.2) Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ be differentiable at the point $z_0 = x_0 + iy_0$. **Prove** that the **partial derivatives of** u and v exist at the point (x_0, y_0) and satisfy the equations $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$. **[7]**

2.3) Consider the **harmonic** function $u(x,y) = y^3 - 3x^2y$. Use **2.2)** to **construct** a **harmonic** conjugate $v(x,y)$ of $u(x,y)$ such that $f(z) = u(x,y) + iv(x,y)$ is **analytic**. **[6]**

[20]

QUESTION 3

3.1) Compute $\int_C \frac{2z-1}{z^2-z} dz$ where $C: |z| = \frac{1}{2}$. [**Hint:** Use $\int_C \frac{1}{z-z_0} dz = 2\pi i$, where C **is a simple closed contour** with positive orientation such that z_0 **lies interior to** C together with the **Cauchy-Goursat theorem**.] **[6]**

3.2) Use the **Cauchy Integral Formula** to compute $\int_C \frac{\sin(z)}{z^2+1} dz$ along the positive oriented contour $C: |z - i| = 1$. [6]

3.3) Use the **Cauchy Integral Formulae for derivatives** to compute $\int_C \frac{1}{(z^2+1)^2} dz$ along the positive oriented contour $C: |z - i| = 1$. [7]

[19]

QUESTION 4

4.1) Consider $f(z) = \frac{1-z}{z-2} = \frac{z-1}{1-(z-1)}$

4.1.1) Use a **geometric series** to compute the **Taylor series** for $f(z)$ centred about $\alpha = 1$.
4.1.2) Find the **region of convergence** for the series in **4.1.1**. [4,2]

4.2) Compute the **Laurent series** representation for $f(z) = \frac{1}{z^2-5z+6}$ involving **powers of** z in the domain $2 < |z| < 3$. [6]

4.3) Classify the **singularities** (if any) of $f(z) = \frac{\sin(z)}{z^2+z}$. [3]

[15]

QUESTION 5

5.1) Let D **be a simply connected domain** and let C **be a simple closed positively oriented contour** that **lies in** D. If f **is analytic inside** C and **on** C, except at the **points** $z_1, z_2, \dots z_n$ that **lie inside** C, then **prove** $\int_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k)$. [5]

5.2) Use the **theorem** in **5.1** to show that $\int_C \frac{e^z}{z^3+z} dz = 2\pi i(1 - \cos(1))$, where $C: |z| = 2$ [7]

5.3) Use **residues** to compute $\int_{-\infty}^{\infty} \frac{1}{x^2+16} dx$. [5]

[17]

END
