UNIVERSITY OF FORT HARE

Abstract Algebra
MAT 311

Examination

June

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Time: 3 hours

Subject: Abstract Algebra

Maximum Marks: 100

This question paper consists of 5 pages.

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Instructions

Answer all the **nine questions**.

Symbols have the usual meanings.

Question 3 [11]

3.1. Theorem: For $n \geq 2$, the number of even permutations in S_n is the same as the number of odd permutations.

3.1.1. **Prove** the **previously** stated **theorem.**

3.1.2. Find the set of **even permutations** of S_3.

3.1.3. **Verify** that $A_3 \lhd S_3$ by **direct computations, that is,** $\forall g \in S_3, \forall n \in A_3$ **that** $g^{-1}ng = n$. **[5, 2, 4]**

Question 4 [15]

4.1. Theorem: Let G be a group and $H \leq G$. Let $A = \{aH : a \in G\}$ and $B = \{Ha : a \in G\}$. Then $|A| = |B|$.

Proof:

Define $f: A \to B$ by $f(aH) = Ha^{-1}$. If $a_1H = a_2H$ then $a_1 = a_2h$ for some $h \in H$. Therefore $a_2 = a_1h^{-1}$, implying $a_2^{-1} = ha_1^{-1} \in Ha_1^{-1}$. Thus $Ha_2^{-1} = Ha_1^{-1}$ and therefore f **is well defined**. To complete the proof, we must show that f **bijective.**

4.1.1. Complete the **proof** of the previously stated **theorem.** **[4]**

4.2. Theorem: Let G be a finite group and $H \leq G$. Then $|H|$ divides $|G|$.

4.2.1. Use the fact that for any $a \in G$, $|aH| = |H|$ to **prove** the previously **stated theorem.**

4.2.2. Compute the distinct **cosets** of A_4 in S_4.

4.2.3. Use **4.2.2.** to show that $A_4 \lhd S_4$. **[5, 3, 3]**

Question 7 [14]

7.1. Theorem: In the **ring** $\mathbb{Z}_n$, the **zero divisors** are precisely those elements that are **not relatively prime** to $\boldsymbol{n}$.

There are **two parts** to the proof of the **aforementioned theorem**:

1. Let $\boldsymbol{m} \in \mathbb{Z}_n - \{\mathbf{0}\}$ and let $\boldsymbol{d} = \boldsymbol{gcd}(\boldsymbol{m}, \boldsymbol{n}) \neq \mathbf{1}$. Then it can be shown that $\boldsymbol{m}$ **is a zero divisor** by finding a $\boldsymbol{k} \in \mathbb{Z}_n - \{\mathbf{0}\}$ such that $\boldsymbol{km} = \mathbf{0}$.
2. Let $\boldsymbol{m} \in \mathbb{Z}_n - \{\mathbf{0}\}$ and let $\boldsymbol{d} = \boldsymbol{gcd}(\boldsymbol{m}, \boldsymbol{n}) = \mathbf{1}$. Then it can be shown that $\boldsymbol{m}$ **is not a zero in** $\mathbb{Z}_n$.

7.1.1. **Write** out the **proof** for **part 1** of the previously stated theorem.

7.1.2. What can be **deduced** from the **theorem in 7.1** about **zero divisors** in $\mathbb{Z}_p$ ($\boldsymbol{p}$ is a prime).

7.1.3. Find all **zero divisors** in $\mathbb{Z}_{12}$.

7.1.4. Do the **cancellation laws** hold in $\mathbb{Z}_{12}$? **Explain.**

7.1.5. **Explain** why $\mathbb{Z}_{12}$ is **not** an **integral domain.**

7.1.6. Find the **characteristic** of $\mathbb{Z}_{12}$. **[5, 2, 2, 2, 2, 1]**

Question 8 [15]

8.1. Fill in the **missing words** below.

i. Let $\mathrm{I} \neq R$ be an ideal of a ring R and whenever J is an ideal such that $\mathrm{I} \subseteq J \subseteq R$ then $\mathrm{I} = J$ or $J = R$. Then I is called a ________ ideal of R.

ii. If R is a commutative ring with unity, and $a \in R$, then the set $\mathrm{I} = \{ra : r \in R\} = \langle a \rangle$ = is the ______ ideal of R generated by a.

iii. Let R be a commutative ring. An ideal $\mathrm{I} \neq R$ is a ______ ideal if $ab \in \mathrm{I}$, for $a, b \in R$, implies $a \in \mathrm{I}$ or $b \in \mathrm{I}$. **[1,1,1]**