UNIVERSITY OF FORT HARE

MAT 223

SUPPLEMENTARY EXAMINATIIONS

JANUARY 2025

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Time: 3 HOURS

Subject: Linear Algebra

Marks: 100

This paper consists of 3 pages including cover page

Internal Examiner

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Instructions

All questions may be answered

Show all your working

Sloppy work will be penalized

Question 1

1. Find the value of k for which the system has no solution. [7]

$$\begin{aligned} x+2y-3z &= 4 \\ 3x-y+5z &= 2 \\ 4x+y+(k^2-14)z &= k+2. \end{aligned}$$

2. Let $A = \begin{bmatrix} 2 & 2 & -1 \\ 4 & 0 & 4 \\ 3 & 4 & 4 \end{bmatrix}$. Express A as a product of elementary matrices. [7]

3. Let $\vec{v}$ and $\vec{w}$ be vectors in a vector space V. Prove that $\|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$. [5]

4. Decide whether or not the set $\mathbb{R}^2$, with addition defined by $(x,y) \dagger (a,b) = (x+a+1, y+b)$, and with the usual scalar multiplication, is a *vector space*. [7]

Question 2

1. Let $A = \begin{bmatrix} \frac{2}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{3}{4} \end{bmatrix}$, $C = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$. Without computing C^{-1}, verify that $C^{-1}AC = \begin{bmatrix} \frac{5}{12} & 0 \\ 0 & 1 \end{bmatrix}$ [6]

2. Use the casting-out technique to find a basis for the subspace $W = \mathrm{sp}(x^2-1, x^2+1, 3, 2x-1, x)$ of the vector space of polynomials. [7]

3. In the vector space M_{22}, let the inner-product be defined by

$$\left\langle \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \right\rangle = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4.$$

Determine whether or not this definition on M_{22} satisfies all the conditions of an inner-product. [7]

4. Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be a linear transformation for which

$$T(1+x) = 1+x^2,\ T(x+x^2) = x-x^2,\ T(1+x^2) = 1+x+x^2.$$

Find $T(4-x+3x^2)$. [7]

Question 3

1. State whether the following statements are True or False. If false, state the true version of the statement: [15]

   (a) A subset W of a vector space V is said to be a transformation of V if W itself a vector space under the operations on V.

   (b) A finite list of nonzero vectors in a vector space V is linearly independent iff no vector in the list is equal to a linear combination of its predecessors.

   (c) Every line in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$ generated by a single vector.

   (d) If two rows of a square matrix A are equal, then the determinant of A is undefined.

   (e) If λ is an eigenvalue of an invertible matrix A with $\vec{v}$ as a corresponding eigenvector, then $-\lambda$ is an eigenvalue of A^{-1} again with $\vec{v}$ as a corresponding eigenvector.

2. Let P, Q and R be $n \times n$ matrices. Recall that P is **similar** to Q (denoted by $P \sim Q$) if there exists an invertible $n \times n$ matrix C such that $C^{-1}PC = Q$. Prove that $\sim$ is an **equivalence relation.** [7]

3. Let $A = \begin{bmatrix} 0 & 1 \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}$.

   (a) Diagonalize the matrix A. [7]

   (b) Hence, find an expression for A^k involving the matrix of eigenvectors of A. [5]

4. Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$ and $\vec{u} \neq \vec{0}$. With the aid of a diagram, define what is meant by the *projection* of $\vec{v}$ onto $\vec{u}$. [4]

5. Suppose $\vec{u}, \vec{v}$ and $\vec{w}$ are vectors in an inner product space such that

$$\langle \vec{u}, \vec{v} \rangle = 1,\ \langle \vec{u}, \vec{w} \rangle = 5,\ \langle \vec{v}, \vec{w} \rangle = 0,\ ||\vec{u}|| = 1,\ ||\vec{v}|| = \sqrt{3},\ ||\vec{w}|| = 2.$$

   Use the above information to evaluate

   (a) $\langle \vec{u} + \vec{w}, \vec{v} - \vec{w} \rangle$ [4]

   (b) $||\vec{u} + \vec{v}||$ [5]

END OF EXAMINATION