UNIVERSITY OF FORT HARE

MAT 212

DEGREE EXAMINATIONS

June 2023

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Time: **3 HOURS**

Subject: FUNDAMENTALS

Marks:100

This question paper consists of 3 pages

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Instructions

Answer **any 4** questions **ONLY**.
Symbols have the usual meanings

Question One [25 marks]

1.1 For propositions P, Q and R, prove that $P \vee (Q \wedge R)$ is logically equivalent to $(P \vee Q) \wedge (P \vee R)$ (6)

1.2 Using deductive reasoning, show that the conjecture: $\left| \int_1^\infty \frac{\cos(x)}{x^2} dx \right| < \infty$ is true. (5)

1.3 Let P be a statement, T a tautology, and C a contradiction. Prove that:

(i) $P \vee T$ is a tautology. (3)

(ii) $P \wedge C$ is a contradiction. (3)

1.4 Briefly explain how a theorem differs from a

(a) Corollary (2)

(b) Conjecture (2)

(c) Axiom (2)

(d) Lemma (2)

Question Two [25 marks]

2.1 Use Proof by Contradiction to prove the following theorem:

Theorem: For $p > 3$, there are no triples of prime numbers of the form $(p,\ p+2,\ p+4)$. (7)

2.2 Prove the following theorem using the method of proof by contrapositive.

Theorem: Let x and y be positive real numbers. If $x \neq y$, then $\ln(x) \neq \ln(y)$. (6)

2.3 Negate each of the following mathematical sentences:

$\forall n$ in the natural numbers, $n^2 - n$ is an even number. (2)

$\exists r \in (-\infty, \infty) \ni: 71 = 3r + 2$ (2)

$\forall i \in \{1, 2, 3, \cdots, 10\}, 2^{-i} \geq 0.0005$ (2)

2.4 Write the following statements using the existential and universal quantifiers where appropriate:

(a) For every positive real number x, $x^3 - 2x^2 + x > 0$. (2)

(b) There exists a real number x, such that $x^2 - 3x = 4$. (2)

(c) For every positive real number ε, there exists a positive number ∂ such that $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \partial$ (2)

Question Three [25 marks]

3.1 Prove that there exist irrational numbers a and b such that a^b is rational. (5)

3.2 Given sets A, B, and C, show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$ (6)

3.3 Let m and n be positive integers.

(i) What is $m \cap n$? (3)

(ii) What is $m \cup n$? (3)

3.4 Let A be a set. Does $A \times A = A$? Justify your answer. (4)

3.5 Which of these statements is true? Justify your answer.

(a) $(\forall x: \mathbb{Z}) x \geq 0$ (2)

(b) $(\forall x: \mathbb{N}) x \geq 0$ (2)

Question Four [25 marks]

4.1 Prove that if A and B are sets, then $P(A) \cup P(B) \subseteq P(A \cup B)$. (6)

4.2 Let $A = [-2,\ 5]$ and $B = (0,\ 10]$. Find: (i) $A - B$ (ii) $A \Delta B$ (3, 3)

4.3 Let $A = \{1,\ 2,\ 3\}$. Let $F: A \rightarrow PP(A)$ be defined by requiring that $F(n) = \{B \in PA | n \in B\}$. What are $F(1)$ and $F(2)$? (3, 3)

4.4 Show that $\{1, 2, 3\}$ under multiplication *modulo* 4 is not a group but that $\{1, 2, 3, 4\}$ under multiplication *modulo* 5 is a group. (5)

4.5 Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$. Find $A \times B$. (2)

Question Five [25 marks]

5.1 Prove the following biconditional theorem:

Theorem: Let $m, n \in \mathbb{N}$. Then $m + n$ is odd if and only if exactly one of m and n is odd. (6)

5.2 List the elements of these sets.

(a) $\{n - 1 \in \mathbb{Z} | n \text{ divides } 12\}$ (2)

(b) $\{n^2 \in \mathbb{N} | n \text{ divides } 12\}$ (2)

(c) $\{n^2 \in \mathbb{Z} | n \text{ divides } 12\}$ (2)

5.3 Reword the following predicates so that they do begin with "¬", and determine their true value. x is real number.

(a) $\neg(x < 10) \wedge (x > 12)$ (3)

(b) $\neg(x < 10) \wedge (x < 12)$ (3)

(c) $\neg(\neg(x > 5) \wedge \neg(x < 6))$ (3)

5.4 Define the following terms:

(i) Group (2)

(ii) Ring (2)

END