UNIVERSITY OF FORT HARE

Complex Analysis

MAT 322

Degree Examinations
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Subject: MAT 322

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This question paper consists of 3 pages including the cover page

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Instructions

Answer all questions.
Symbols used have the usual meaning

Question One

1.1Let $z = x + iy$ be a complex number. Sketch the set of points determined by the relation $\operatorname{Re}(z+1) = 0$. (3)

1.2 Show that $\arg(z_1 z_2 z_3) = \arg z_1 + \arg z_2 + \arg z_3$ for any $z_1, z_2, z_3 \in \mathbb{C}$. (3)

1.3 Determine whether the following point lie inside the circle $|z - i| = 1$

$$-\frac{1}{2} + i\sqrt{3}$$ (2)

1.4 Define three elementary mappings transforming $\mathbb{C}$ into itself (4)

1.5 Show that the image of the open disk $|z - 1 - i| < 1$ under the linear transformation $w = (1-2i)z + 3 + i.$ is the open disc $|w - 6| < \sqrt{5}$ (3)

Question Two

2.1 Let a complex function be f given by $f(z) = u(x,y) + iv(x,y)$ and be differentiable at the point $z_0 = x_0 + iy_0$. Show that the first partial derivatives of u and v at the point (x_0, y_0) exist and verify the Cauchy-Riemann equations that is

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \text{ and } \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0).$$ (5)

2.2 Show that $f(z) = \frac{\bar{z}}{z}$ does not have a limit as $z \to 0$. (3)

2.3 Show that if f given by $f(z) = u(x,y) + iv(x,y)$ is analytic in the domain D, then the functions u and v are both harmonic in D.
Hence if u is described by $u(x,y) = \sin y \sinh x$, show that it is harmonic and determine the harmonic conjugate v of u such that $f(z) = u + iv$ is analytic. (5)

2.4 (a) Given a complex –valued function $f : [a,b] \to \mathbb{C}$, how is its integral $\int_a^b f(t)dt$ defined? (b)Hence show that $\int_0^{\frac{\pi}{4}} te^{it}dt = \frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2} - 1 + i\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}\pi}{8}\right)$ (7)

2.5Let C be the line segment from 2 to $1+i$, parameterize it and use such parameterization to find the contour integral of $\int_C \exp z dz$ (7)

Question Three

3.1 Evaluate $\int_C (2z-1)(z^2-z)^{-1}dz$ where $C:|z-1|=2$ with positive orientation. (4)

3.2State and prove the Cauchy-Goursat Theorem. (7)

3.3State and prove the Cauchy-Integral Formula . Hence use it to show that $\int_C (\exp z+\cos z)z^{-1}dz = 4\pi i$ where C is a circle $|z|=1$ with positive orientation.(10)

Question Four

4.1 Show that $\sum_{n=0}^{\infty}\frac{(1-i)^n}{2^n} = 1-i$ (6)

4.2 State without proof Taylor s Theorem. (4)

4.3 Find two Laurent Series for $f(z)=z^{-1}(4-z)^{-2}$ involving powers of z. (5)

4.4 Given $f(z)=(3z^4+10z^2+3)^{-1}$.Locate the poles of f and their orders. (4)

4.5 (a) Define a singular point of a complex valued function f. (2)

(b) Hence prove that if $f(z)$ has a removable singularity at z_0, then $\frac{1}{f(z)}$ has either a removable singularity or a pole at z_0. (4)

4.6 The residue of f at z_0 is denoted by $\operatorname{Res}[f,z_0]$. Give a definition of this residue and hence use this theory to show that $\operatorname{Res}[f,z_0]=\operatorname{Res}[f,0]=2$ for $f(z)=\exp(\frac{2}{z})$. (5)

4.7 State and prove Cauchy s residue Theorem. Hence find $\int_C \frac{\sin z}{4z^2-\pi^2}dz$ where C is the circle $|z|=2$ with positive orientation. (7)