UNIVERSITY OF FORT HARE

STM 211 & 212

DEGREE EXAMINATIONS
JUNE 2023

....

TIME : 3 HOURS

SUBJECT : INTRODUCTION TO MATHEMATICAL STATISTICS A & B

MARKS : 100

This **paper** consists of 5 printed pages, including the cover page

Internal Examiners

Mr. I. Jubane

Moderator

Dr. Mutambayi

Instructions

Answer all questions

Statistical Tables will be provided

Question One **19 Points**

1.1 A simple binary communication channel carries messages by using only two signals, 0 and 1. For a given binary channel, 40% of the time, a 1 is transmitted; the probability that a transmitted 0 is correctly received is 0.90, and the probability that a transmitted 1 is correctly received is 0.95.

1.1.1 Draw a tree diagram to represent this information [2]

1.1.2 Determine (a) the probability of a 1 being received, [3]

1.1.3 Given a 1 is received, determine the probability that 1 was transmitted. [3]

1.2 Suppose that four balls are selected one at a time, without replacement, from a box containing 4 red balls and 6 blue balls. Determine the probability of obtaining the sequence of outcomes red, blue, red, blue. [3]

1.4 Of six employees, three have been with the company for five or more years. If four employees are chosen randomly from the group of six, the probability that exactly two will have five or more years of seniority is [3]

1.5 Three plants, C_1, C_2, and C_3, produce respectively, 10%, 50%, and 40% of a company's output. Although plant C_1 is a small plant, its manager believes in high quality and only 1% of its products are defective. The other two, C_2 and C_3, are worse and produce items that are 3% and 4% defective, respectively. All products are sent to a central warehouse. If one item is selected at random and observed to be defective, calculate the probability that the item comes from plant C_1. [5]

Question Two — 17 Points

2.1 Let the random variable X of the discrete type have the *pmf* given by the table

X	0	1	2	3
Probability	0.216	0.432	0.288	0.064

2.1.1 Calculate $\Pr(0 < X \leq 2)$ [2]

2.1.2 Find the mean of X and variance of $Y = 2X^2 + X - 1$. [5]

2.2 Suppose that a random variable X has a discrete distribution with the following function:

$$f(x) = \begin{cases} k2^{1-x} & \text{for } x = 1,2,3,\ldots, \\ 0 & \text{otherwise.} \end{cases}$$

2.2.1 Find a number k such that $f(x)$ would be a probability function. [4]

2.2.2 Calculate $\Pr(X > 1)$ [3]

2.2.2 Show that $E[X] = \frac{1}{k}$ [3]

Question Three	**24 Points**

3.1 A cubic die is thrown n times. Let X_i is the number of spots on the i^{th} throw and Y_n be the total number of spots shown.

3.1.1 Show that $E[Y_n]=7n/2$ and $\text{Var}[Y_n]=35n/12$. [6]

3.1.2 State Chebyshev's inequality and find an n such that. [6]

$$\Pr\left(\left|\frac{Y_n}{n}-3.5\right|>0.1\right)=0.1.$$

3.1.2 Let $\lambda=np$.
Show that

$$\lim_{n\to\infty}\binom{n}{y}\frac{\lambda}{n}^{y}(1-\frac{\lambda}{n})^{n-y}=\frac{\lambda^y e^{-\lambda}}{y!},x=0,1,2,\ldots$$

and state the precise relation between the binomial and Poisson distributions. [6]

3.2 $X_1,\ldots,X_n$ are independent, identically distributed random variables with mean μ and variance σ^2. Find the mean of

$$S_{XX}=\sum_{i=1}^{n}(X_i-\bar{X})^2 \text{ where } \bar{X}=\frac{1}{n}\sum_{i=1}^{n}X_i.$$ [6]

Question Three **40 Points**

4.1 The joint probability distribution function of the random variables X and Y is given by:

$$f(x,y)=\begin{cases}24(1-x)y, & 0<y<x<1 \\ 0 & \text{elsewhere}\end{cases}$$

4.1.1 Check that f is a proper PDF (i.e. its integral equals 1). [3]

4.1.2 Find the marginal distributions of X and Y. [6]

4.1.3 Hence, calculate $\Pr\ Y \leq \frac{1}{3} \big| X = \frac{2}{3}$. [4]

4.1.4 Show that the $\operatorname{cov}(X,Y) = \dfrac{2}{75}$. [5]

4.2 Let X be an exponentially distributed random variable with PDF

$$f(x)=\lambda e^{-\lambda x}, x>0,$$

where $\lambda > 0$.

4.2.1 Find the PGF for X and compute its mean and variance. [6]

4.2.2 Calculate :

4.2.2.1 The first quartile [2]

4.2.2.1 The second quartile [2]

4.2.2.1 The third quartile [2]

4.2.3 What is the mean and variance of X_n? [5]

4.2.3 derive the distribution of the random variable $Y = e^{-x}$. [5]