UNIVERSITY OF FORT HARE

General Topology
MAT 504

Degree Examinations

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Time: 4 Hrs

Subject: MAT 504

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This question paper consists of 2

Internal examiner

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Instructions

Answer all questions.

Symbols used have the usual meaning

Question One

1.1 Let X be non-empty set and consider the power set of X, $P(X)$. When is a family $\Im \subset P(X)$ said to be a *topology* on X. [3]
Hence if $X = \{a,b,c\}$ and $\Im = \{\phi, \{a\}, \{c,b\}, \{b\}. \{a,b,c\}\}$. Is $\Im$ a topology on X? [3]

1.2 Given a topological space $(X,\Im)$ and consider $x \in X$, when is a subset $\xi \subset N_X(x)$ said to be a *fundamental system* of x. [2]

1.3 Give the definition of (i) *Hausdorff topological space.*
(ii) *Regular topological space.* [2]

1.4 Prove that the interior of a subset A, $\text{int}(A)$ in $(X,\Im)$ is an open set. [3]

1.5 Let X be a set and $B \subseteq P(X)$. Prove that B is a *basis for a topology* on X (i.e $\{\bigcup_{S \subset C} S \mid C \in B\}$ a topology on X) iff $\bigcup_{S \in B} S = X$ and $T, U \in B \Rightarrow T \cap U$ is a union of some elements of B [5]

1.6 Prove that given a topological space $(X,\Im)$ and a subspace $(S,\Im_S)$, a subset B of S is closed in $(S,\Im_S)$ if and only if there exist a closed set F in $(X,\Im)$ such that $B = S \cap F$. [5]

1.7 Prove that given a topological space $(X,\Im)$ and $S \subset X$, then the subset S is *locally closed* if and only if there exist an open set O and a closed set F in $(X,\Im)$ such that $S = O \cap F$. [7]

Question Two

2.1 Given any two topological spaces $(X,\Im)$ and (Y,ς). Give the definition of local continuity of a function $f: X \to Y$ at a point $x \in X$. Hence prove that if $U \subset X$ and $f: X \to Y$ is a mapping then $U \subset f^{-1}(f(U))$. [4]

2.2 Given any two a topological spaces $(X,\Im)$ and (Y,ς), when is a mapping $f: X \to Y$ a homeomorphism? [2]
Hence prove that for such a mapping the following statements are equivalent:

1. f is a homeomorphism of X onto Y.
2. f is continuous and open.
3. f is continuous and closed.
4. $N_Y(f(x)) = \{f(V) : V \in N_X(x)\}$ for any $x \in X$. [8]

2.3 Prove that if $(Y_i, \Im_i)_{i \in I}$ is a family of topological spaces and if (Z,ς) is another topological space and X is equipped with the initial topology with respect to the mapping $f_i : X \to Y_i$,then the mapping $f : Z \to X$ is continuous at $a \in Z$ if and only if $f_i \circ f : Z \to Y_i$ is continuous at $x = a$ for any $i \in I$. [7]

2.4When is the initial topology said to be Hausdorff? Motivate your answer [4]

Question Three

3.1 Prove that a topological space $(X, \Im)$ is Hausdorff if and only if Δ is closed in $X \times X$ where $\Delta = \{(x,x) : x \in X\}$ the diagonal of $X \times X$. [6]

3.2 Let $(Y_i, \Im_i)_{i \in I}$ be a family of topological spaces and $(X, \Im)$ its product topological space. Prove that if $A_i \subset X_i, \forall i \in I$ then $\overline{\prod_{i \in I} A_i} = \prod_{i \in I} \overline{A_i}$. [7]

State any one consequence of this theorem. [2]

3.3 Prove that any point b not belonging to a compact subset A of a Hausdorff topological space $(X, \Im)$ can be separated from the compact subset by open sets. [7]

3.4 Prove that any compact topological space $(X, \Im)$ is regular. [5]

3.5 Let $(X, \Im)$ be a regular topological space, prove that any compact subset that does not intersect with a closed subset of X can always be separated from such a closed set by closed sets. [8]

3.6 Prove that the Cartesian product of any arbitrary family $\{(X_i, \Im_i)\}_{i \in I}$ of compact topological spaces $(X_i, \Im_i)$ is also compact. [8]

Write down a consequence of this theorem. [2]