UNIVERSITY OF FORT HARE

University of Fort Hare
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MATHEMATICAL METHODS & STATISTICAL MECHANICS
PHY322

SUPPLEMENTARY EXAMINATION

OCTOBER/ NOVEMBER

2024

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Time: 3 Hours

Subject: PHY322

Marks: 100

This paper consists of 3 pages including the cover page

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INSTRUCTIONS

Answer all five questions, each question carries 20 marks.

QUESTION 1 [20 MARKS]

1.1 Let $f(x, y, z) = x^2y + ze^{xy}$.

1.1.1 Compute ∇f (4)

1.1.2 Find the direction of greatest change of $f(x, y, z) = x^2y + ze^{xy}$ at $(x, y, z) = (1,1,0)$. (3)

1.1.3 What is the rate of change of $f(x, y, z)$ in this direction? (3)

1.2 Evaluate $\oiint (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{n}\, dS$, where, the vector field $\bar{F} = (2x - y)\hat{\imath} - yz^2\hat{\jmath} - y^2z\hat{k}$ over the upper half of the surface $x^2 + y^2 + z^2 = 1$ bounded by its projection on xy-plane. (10)

QUESTION 2 [20 MARKS]

2.1 Define the Dirac delta function (3)

2.2 2.2.1 Prove that $\delta(g(x)) = \frac{\delta(x - x_i)}{|g'(x_1)|}$ (5)

2.2.2 Use this result to evaluate $\int_{-\infty}^{\infty} \delta(x^2 - 5x - 6)(3x^2 - 7x + 2)dx$. (4)

2.3 Evaluate the following integrals

2.3.1 $\int_0^{\pi} x^2\delta\left(x + \frac{\pi}{2}\right)dx$ (4)

2.3.2 $\int_{-20}^{15} \sin\left(\frac{\pi x}{e}\right)\delta(x^2 - e^2)dx$ (4)

QUESTION 3 [20 MARKS]

3 The Legendre differential equation is given as:

$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$

3.1 Write the Rodrigues's formula which defines polynomial solutions to Legendre differential equation. (3)

3.2 Show that the Legendre polynomials are orthogonal in the interval $[-1; 1]$, that is: $\int_{-1}^{1} P_n(x)P_m(x)dx = 0$ (5)

3.3 Find the first two Legendre polynomials: that is:

3.3.1 $P_0(x)$ (2)

3.3.2 $P_1(x)$ (2)

3.4 3.4.1 Use the Legendre recurrent relation: $(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x)$ to calculate $P_2(x)$ (4)

3.4.2 Express the polynomial $4x^2 + 8x + 6$ in terms of the Legendre polynomials. (4)

QUESTION 4 [20 MARKS]

4.1 The Bessel differential equation is given as: $x^2y'' + xy' + (x^2 - n^2)y = 0$

For n > 0, write the solution to the Bessel differential equation. (3)

4.2 For a particular value of n, the Bessel differential equation is given as:

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

4.2.1 Find the values of n (3)

4.2.2 Using the general solution in (4.1) and the negative value of n calculated (4.2.1) , show that the solution to the equation is: (7)

$$y(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

4.3 Show that: $\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x)$ (7)

QUESTION 5 [20 MARKS]

5.1 State the rank of the following;

5.1.1 Matrix (1)

5.1.2 Vector (1)

5.2 Write the following in tensor notation.

5.2.1 $\vec{A} \bullet \vec{B}$ (1)

5.2.2 $\vec{U} \times \vec{V}$ (2)

5.3 Define Kronecker delta function δ^i_j (2)

5.4 Show that $\delta^i_i = 3$ where i runs from 1 to 3 (3)

5.4 Show that the product of the Levi-Civita tensors

5.4.1 $\in^{ijk}\in_{lmn} = \{\delta^j_m\delta^k_n - \delta^j_n\delta^k_m\}$ for $i = l$ (4)

5.4.2 $\in^{ijk}\in_{lmn} = 2\delta^k_n$ for $i = l$ and $j = m$ (3)

5.5 Show that $\vec{E} = \in^{\mu\alpha\beta}\in_{\beta\gamma\sigma} C_\alpha A^\gamma B^\sigma$ can be written as: $\vec{E} = (\delta^\mu_\gamma\delta^\alpha_\sigma - \delta^\mu_\sigma\delta^\alpha_\gamma)C_\alpha A^\gamma B^\sigma$ (3)