University of Fort Hare

MAT 303

Degree Examinations: November 2018

Subject: Mathematics 3
Paper: Real Analysis

Time: 3 Hours   Marks: 100   Subminimum: 40

This question paper consists of 5 pages

Internal examiner(s)	**External examiner(s)**
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Instructions

Attempt **NO more** than **FIVE (5)** questions. Symbols used have the usual meanings.

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Mathematics Degree Exams November 2018

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Question One

1.1 (a) Define a metric on a non-empty set X. (2)
(b) Define the diameter of a set A in R^n. Hence find the diameter of the set $\{(x,y) : \frac{x^2}{9} + \frac{y^2}{4} \leq 1\}$ in R^2. (3)
(c) Define the trivial metric on X. (1)
(d) Define the usual metric on $\boldsymbol{R}^n$. (1)

1.2 Let d be a metric on X, $A \subset X$ and $p \in X$. The distance between p and A is $d(p, A) = \inf\{d(p, a) : a \in A\}$.
Let d be the trivial metric on $\boldsymbol{R}$ and $A = (0, 5) \subseteq R$. Find $d(0, A)$. If ρ is the usual metric on $\boldsymbol{R}$, find (i) $\rho(0, A)$ and (ii) $\rho(6, A)$. (3)

1.3 Let (X, d) be a metric space. Show that the function ρ on $X \times X$, defined by $\rho(a, b) = \frac{d(a,b)}{1+d(a,b)}$, where a, $b \in X$, is also a metric on X. (4)

1.4 (a) Let $A \subset \boldsymbol{R}^n$. Define what is meant by A is an open subset of $\boldsymbol{R}^n$. (1)
(b) Show that the set $A = \{(x, y) \in R^2 : 0 < y < 3\}$ is open in $\boldsymbol{R}^2$. Illustrate with a sketch. (5)

1.5 Prove that a set $A \subset \boldsymbol{R}^n$ is closed if and only if for every sequence $\{x_k\}$ in A which converges, the limit lies in A. (4)

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Question Two

2.1 Let $B \subset \boldsymbol{R}^n$. Prove that $x \in \overline{B}$ if and only if there is a sequence $\{x_k\}$ in B converging to x, where $\overline{B}$ is the closure of B in $\boldsymbol{R}^n$. (5)

2.2 A metric space (M , d) is complete if every Cauchy sequence in M converges to a point in M. Show that the set $\boldsymbol{Q}$ of all rational numbers (with the usual metric) is not a complete metric space. (2)

2.3 (a) Let $A \subset \boldsymbol{R}$, x, $y \in A$. Define what is meant by a path joining x to y. (1)
(b) Let $A \subset \boldsymbol{R}^n$. Define what is meant by (i) A is compact, (give three equivalent statements),
(ii) A is path-connected. (4)

2.4 Show that $A = \{x \in R^n : ||x|| \leq 5\}$ is path-connected. (3)

2.5 Show that $[0, 1] \cap \boldsymbol{Q}^c$ is not path-connected, where $\boldsymbol{Q}^c$ is the set of all irrational numbers. (3)

2.6 Prove that a set $A \subseteq R$ is connected if and only if A is an interval. (4)

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Question Three

3.1 Let $f : A \to \mathbb{R}^n$. Prove that the following statements are equivalent:
(i) f is continuous on A;
(ii) For each convergent sequence $\{x_n\}$, converging to $x_0 \in A$, the sequence $\{f(x_n)\}$ converges to $f(x_0)$;
(iii) For each open set $U \in R^n$, $f^{-1}(U) \subseteq A$ is open relative to A, i.e. $f^{-1}(U) = V \cap A$ for some open set V in A. (8)

3.2 Let $f : A \to \mathbb{R}^n$ be a continuous function. Prove that
(a) If $K \subseteq A$ and K is connected, then $f(K)$ is connected. (4)
(b) If $B \subseteq A$ and B is compact, then $f(B)$ is compact. (3)

3.3 Let $f : [0, 1] \to [0, 1]$ be continuous. Show that f has a fixed point. (3)

3.4 (a) Find a continuous map $f : \mathbb{R} \to \mathbb{R}$ and a compact set $K \subseteq \mathbb{R}$ such that $f^{-1}(K)$ is not compact (2)
(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and let $A = \{f(x) : ||x|| = 2\}$. Show that A is a closed interval. (3)

3.5 Let $f : A \to \boldsymbol{R^m}$ be continuous and let $K \subseteq A$ be a compact set. Prove, from definition, that f is uniformly continuous on K. (4)

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Question Four

4.1 Let $\{f_k : A \to R^m\}$ be a sequence of functions.
(a) Define what is meant by the sequence of functions converges **uniformly** to f. (2)
(b) Let $\{f_k : A \to \boldsymbol{R^m}\}$ be a sequence of continuous functions, and suppose $f_k \to f$ **uniformly**. Prove that f is continuous. (4)

4.2 (a) State, without proof, the Weierstrass M-test for uniform convergence. (2)
(b) Show that $\sum_{n=1}^{\infty} \frac{(\cos(nx))^2}{n^2}$ converges uniformly. (2)

4.3 Let $f_n(x) = \frac{x}{nx+1}$ on $(0, 1)$. Discuss uniform convergence of the sequence $\{f_n\}$. (3)

4.4 Let C_b denote the space of bounded continuous functions $f : A \subseteq \boldsymbol{R^n} \to \boldsymbol{R^m}$.
Show that C_b is a Banach space, (i.e. a complete normed vector space). (4)

4.5 (a) Let $f : A \subseteq \boldsymbol{R^n} \to \boldsymbol{R^m}$. Define what is meant by f is differentiable at $x_0 \in A$. (2)
(b) Define the Jacobian matrix of the function f in (a) above. (3)
(c) Let $f : \boldsymbol{R^2} \to \boldsymbol{R^3}$ be given by $f(x, y) = (x^2, x^3y, x^4y^2)$. Compute the Jacobian matrix of f. (2)

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Question Five

5.1 Let $x,\ y \in R$. Let $U(x,y) = \frac{x^4+y^4}{x}$ and $V(x,y) = \cos x + \sin y$. Find at least one point near which we can solve for x, y in term of U, V. (4)

5.2 Let $x,\ y \in R$. Let $U(x,y) = e^x \sin y$ and $V(x,y) = e^x \cos y$. Show that $f(x,y) = (U(x,y), V(x,y))$ is locally invertible, but NOT invertible. (4)

5.3 (a) State the Implicit Function Theorem. (2)
(b) Use the Implicit Function Theorem to discuss the solvability for u, v, w in terms of x, y, z near $x = y = z = 0,\ u = v = 0,\ w = -2$ in the system
$3x + 2y + z^2 + u + v^2 = 0$
$4x + 3y + z + u^2 + v + w + 2 = 0$
$x + y + w + u^2 + 2 = 0.$ (4)

5.4 (a) Define a partition of an interval $[a, b]$. (b) Define the upper and the lower sums of a function $f : [a, b] \to [0, \infty)$. (c) Define what is meant by f is Riemann integrable. (5)

5.5 State, without proof, Riemann's condition for integrability. (2)

5.6 Use Riemann's condition to prove that a continuous function $f : [a, b] \to R$ is Riemann integrable. (6)

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Question Six

6.1 Use the Riemann's condition to show that $\int_0^1 x dx$ exists and equals $\frac{1}{2}$. (6)

6.2 (a) Define the volume of a subset A of R^n. (b) If $A = [3, 5] \times [-1, 2]$ and $n = 2$, find the volume of A. (c) Define what is meant by a subset A of R^n has measure zero. (4)

6.3 (a) State, without proof, Lebesgue's Theorem. (b) State a corollary of the Lebesgue Theorem that characterises a set of measure zero in terms of volume. (4)

6.4 Let $f : A \to \mathbb{R}$ be defined by $f(x, y) = \begin{cases} x^2 + \sin(1/y) & \text{if } y \neq 0 \\ x^2 & \text{if } y = 0 \end{cases}$ where $A = \{(x, y) \mid x^2 + y^2 < 1\}$. Show that f is Riemann integrable. (4)

6.5 Show that $\int_0^\infty e^{-x} x^{p-1} dx$ converges for all $p > 0$. (4)

6.6 Let A be a set and $\mathcal{P}(A)$ the power set of A. Prove that $|A| < |\mathcal{P}(A)|$. (4)

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