UNIVERSITY OF FORT HARE

Partial Differential Equations MAP 321

Degree Examinations

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Time: 3 Hrs

Subject: Applied Mathematics 3

Marks: 100

This question paper consists of 3 pages

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Instructions

Answer **ANY 4** questions.

Symbols have the usual meanings

Question 1

1.1 Consider the following Cauchy equation $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + \sqrt{2}z = 0$ for $z(x,0) = x + e^x$ and use the method of rotation of axis to find its

(a) General solution and

(b) Its particular solution. (6, 3)

1.2 Use the method of characteristics to find the general solution to the diff equation $xZ_x + yZ_y = Z^2$.(7)

1.3 Find the general solution to the following differential equation

$$\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x \sin y.$$ (9)

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Question 2

2.1 Describe the regions in the x-y plane where the following differential equation is elliptic, parabolic or hyperbolic; $y\frac{\partial^2 U}{\partial x^2} - 2\frac{\partial^2 U}{\partial x \partial y} + e^x\frac{\partial^2 U}{\partial y^2} + x\frac{\partial U}{\partial x} - U = 0.$ (6)

2.2 Use Laplace transforms to solve the following PDE,

$$\frac{\partial U}{\partial t} - 3\frac{\partial^2 U}{\partial x^2} = 0, \; where\; 0 < x < 3, \; t > 0,$$

subject to $U(0,t) = U(3,t) = 0,$

$$U(x,0) = 10 \sin 2\pi x - 6 \sin 4\pi x$$ (9)

2.3 Use the method of separation of variables to show that the following boundary value problem,

PDE $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, where\; 0 < x < \pi, \; 0 < y < b,$

subject to $U(0,y) = U(\pi,y) = 0, 0 < y < b,$

$U(x,b) = 0, \quad U(x,0) = 3 \sin x, \; 0 < x < \pi$

has a particular solution, $Z(x,y) = \frac{3 \sin x \sinh(b-y)}{\sinh b}$. (10)

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P.T.O

Question 3

3.1 Prove that a necessary condition for

$$w = f(z) = u(x, y) + iv(x, y)$$

to be analytic in a region is that the Cauchy equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

be satisfied in the region. (6)

3.2 If $v(x, y) = 2x(1 - y)$, find $u(x, y)$ the harmonic conjugate such that $f(z) = u(x, y) + v(x, y)$ is an analytic function of z. (4)

3.3 Prove that, if $f(z)$ is analytic inside and on a simple closed curve C then $\oint_C f(z)dz = 0$. (3)

3.4 Evaluate the integral $\int_{|z-1|=\frac{3}{2}} \frac{z^3+1}{(z+1)^2(z^2+1)} dz$ using the Cauchy's integrals formulas. (5)

3.5 Evaluate the following real definite integral $\int_0^{2\pi} \frac{cos^2\,\theta}{6-2\,sin\,\theta} d\theta$ by contour integrations. (7)

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Question 4

4.1 Find the real and imaginary parts of $f(z) = \frac{i(z-1)}{1+z}$ in terms of x and y . (4)

4.2 Define a conformal mapping $w = f(z)$ at a point z. (1)

4.3 Prove that (i) $w = z + \beta$ represents a translation (3)

(ii) $w = az$ represents a stretching or contraction for a real number. (3)

4.4 Consider a square $S = \{x + iy | 0 < x.y < 2\}$,

(i) Draw a rough sketch showing the square on its plane. (1)

(ii) Find the image of the square in (i) above under the map $f(z) = z^2$ and sketch it. (6)

4.5 Show that the function $e^{-x} \sin y$ is harmonic. (2)

4.6 Show that this function $e^{-x} \sin y$ remains harmonic under the transformation, $z = w^2$. (5)

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Question 5

5.1 Show that the Euler equation $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$ can also be written in the form

$$\frac{d}{dx}\left[F - y'\frac{\partial F}{\partial y'}\right] - \frac{\partial F}{\partial x} = 0.$$ (3)

5.2 If F does not involve x explicitly, show that the Euler equation can be integrated to yield

$$F - y'\frac{\partial F}{\partial y'} = c.$$

(3)

5.3 Find (a) the extremals of $\int_{x_1}^{x_2} \sqrt{1 + (y')^2}dx$ and thus (b) show that the shortest distance between the two points in a plane is a straight line. (5, 4)

5.4 Derive the equation for the geodesic on the surface of a sphere with radius, R. (10)

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