UNIVERSITY OF FORT HARE

MAT 303

SUPPLEMENTARY EXAMINATIIONS

January 2024

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Time: 3 HOURS

Subject: Real Analysis

Marks: 100

This paper consists of 3 pages including cover page

Internal Examiner

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Instructions

All questions may be answered

All symbols used have their usual menaing

Sloppy work will be penalized

Question 1

1. Define precisely what it means for a set A to be *countable*. [3]

2. State the *Cantor-Schröder-Bernstein Theorem.* [2]

3. Let A and B be sets such that $B \subseteq A$. Prove that if there is an injective function $f : A \to B$ then there is a bijective function $g : A \to B$. [6]

4. Show that a *union* of an arbitrary collection of open sets in $\mathbb{R}$ is an open set. [6]

5. (a) Define what is meant by a compact set. [3]
   (b) Prove that a subset K of $\mathbb{R}$ is closed and bounded if K is compact. [7]

6. (a) State the *Monotone Convergence Theorem.* [3]
   (b) Hence show that $(\frac{n+1}{n})$ is a monotone convergent sequence. [6]

Question 2

1. State and prove Bolzano-Weierstrass Theorem for sequences. [7]

2. Let $a, L \in \mathbb{R}$ and let $f : D \to \mathbb{R}$ be a function acting in some interval about a, except possibly at a .
   (a) State precisely what is meant by the limit L of f at a. [3]
   (b) Use the definition in (a) to show that the function $f = \dfrac{2x+3}{x+2}$ has a limit 1 at -1. [5]

3. For $L_1, L_2 \in \mathbb{R}$, suppose that $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$, where $f, g : I \to \mathbb{R}$. Prove that $\lim_{x\to a}[f(x) + g(x)] = L_1 + L_2$. [6]

4. (a) Define what is meant by a *uniformly continuous* function f [3]
   (b) Hence, show that the function $f(x) = \dfrac{1}{x}$ is uniformly continuous on $[a, \infty)$, for a positive a. [5]

Question 3

1. (a) For Riemann Integration, define precisely what is meant by the *upper* and the *lower sum* of a function f relative to the partition P, denoted by $U(f, P)$ and $L(f, P)$, respectively. [6]
   (b) Define what is meant by a *refinement* of a partition. [2]

2. Suppose f is a real-valued function which is bounded on $[a, b]$ and that P^* is a refinement of a partition P of $[a, b]$. Prove that $L(f, P) \leq L(f, P^*)$. [7]

3. (a) Let f be a real-value function on an interval $[a, b]$. Define what is meant by the upper and lower integral of f. [3, 3]

   (b) Hence, state and prove the relation between the upper and lower integral of f. [7]

4. Let f and g be integrable functions on $[a, b]$, and let $\mathcal{R}[a, b]$ be a family of all Riemann-integrable functions on $[a, b]$. Prove that $f + g \in \mathcal{R}[a, b]$, that $\int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)dx + \int_a^b g(x)dx$, [7]

END OF EXAMINATION