UNIVERSITY OF FORT HARE

Real Analysis
MAT 244

Examination

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Time: 3 hours

Subject: Real Analysis

Maximum Marks: 100

This question paper consists of 5 pages

Internal examiner

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Instructions

Answer **Five Questions**

Symbols have the usual meanings.

QUESTION 1

Choose at **least one correct answer** and **write** your **answers** on the **question paper**.

1.1) Let $S = \{x \in \mathbb{R} | x + |x+1|\}$. Determine the $\boldsymbol{sup}(S)$ and $\boldsymbol{inf}(S)$.

(A) $sup(S) = 0$ and $inf(S) = 1$. **(B)** $\mathbf{sup}(S) = \mathbf{1}$ and $\mathbf{inf}(S)$ does not exist.
(C) $sup(S)$ does not exist and $inf(S) = -1$ **(D)** $\mathbf{sup}(S) = -1$ **and** $\mathbf{inf}(S)$ **does not exist.**
(E) None of the above.

1.2) Choose the **incorrect statement.**

(A) Between any two distinct real numbers is a rational number.
(B) Between any two distinct real numbers is an irrational number.
(C) If $a, b \in \mathbb{R}$ and $b > 0$ then $\exists n \in \mathbb{N} \ni nb > a$.
(D) There is a real number between any two consecutive integers.
(E) None of the above.

1.3) Determine which **set** below is **not open**.

(A) $\cup_{n=1}^{\infty}(-n, n)$ **(B)** $\cap_{n=1}^{\infty}\left(-\frac{2}{n}, \frac{2}{n}\right)$, **(C)** $\cap_{n=1}^{\infty}\left(0, \frac{1}{n}\right)$, **(D)** $\cap_{n=1}^{5}(-n, n)$,
(E) None of the above.

1.4) Find the set of **accumulation point(s)** of $S = \left\{\frac{n-1}{n} | n \in \mathbb{N}\right\}$.
(A) $\{-1\}$ **(B)** $S \cup \{-1\}$ **(C)** $\{1\}$. **(D)** $S \cap \{-1\}$ **(E)** None of the above.

1.5) Choose the **correct statement.**

(A) The sequence $\{1 + (-1)^n\}$ is bounded.
(B) Since $\{(-1)^n\}$ oscillates between 1 and -1 it implies $\left\{\frac{(-1)^n}{n}\right\}$ is divergent.
(C) Since $|\cos(n\pi)| \leq 1$ it implies that $\{\cos(n\pi)\}$ is divergent.
(D) The sequence $\left\{\sin\left(\frac{n\pi}{2}\right)\right\}$ is bounded and convergent.
(E) None of the above.

1.6) Find a **monotone increasing sequence** below.

(A) $\{1 + (-1)^n\}$ **(B)** $\left\{\frac{1}{n}\right\}$ **(C)** $\left\{\frac{n+1}{n+2}\right\}$ **(D)** $\{2n - |10 - n|\}$ **(E)** None of the above.

1.7) Let $f(x) = x^4 - x^2$.

(A) Since f is continuous on $[0,1]$, differentiable on $(0,1)$ and $f(0) = f(1) = 0$ by

Rolle's theorem $\exists c \in (-1,0) \ni f(c) = 0$.

(B) Since f is continuous on $[0,1]$, differentiable on $(0,1)$ and $f(0) = f(1) = 0$ by Rolle's theorem $\exists c \in (-1,0) \ni f'(c) = 0$.

(C) Since f is continuous on $[-1,0]$, differentiable on $(0,1)$ and $f(0) = f(-1) = 0$ by Rolle's theorem $\exists c \in (-1,0) \ni f'(c) = 0$.

(D) Since f is continuous on $[-1,0]$, differentiable on $(0,-1)$ and $f(0) = f(-1) = 0$ by Rolle's theorem $\exists c \in (-1,0) \ni f'(c) = 0$.

(E) None of the above.

1.8) The series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ is

(A) conditionally convergent **(B)** absolutely convergent **(C)** divergent **(D)** None of the above.

1.9) By the **ratio test** $\sum_{n=1}^{\infty} \frac{n}{5n-1}$

(A) diverges since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 5$.

(B) converges since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{5}$.

(C) converges since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 5$.

(D) diverges since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ and hence the test is inconclusive.

(E) None of the above.

1.10) The set $\left\{\frac{n+1}{n} | n \in \mathbb{N}\right\}$ is

(A) not bounded and contains all its accumulation point and hence compact.

(B) bounded and contains all its accumulation point and hence compact.

(C) bounded and does not contain all its accumulation point and hence is not compact.

(D) bounded and contains all its accumulation point and hence compact.

(E) None of the above.

[10x3]=[30]

QUESTION 2

2.1) Prove that $\mathbb{N} \sim E$, where E is the set of **even natural numbers**.

2.2) Let $S \subseteq R$ and S is nonempty. **Prove**, if $b = sup(S)$ **then b is an upper bound** of S and for $\varepsilon > 0, \exists x \in S$ such that $x > b - \varepsilon$.

2.3) Let $a, b \in \mathbb{R}$. **Prove** that $|\boldsymbol{a} + \boldsymbol{b}| \leq |\boldsymbol{a}| + |\boldsymbol{b}|$.

2.4) Prove $S = (0, \infty)$ is an **open** subset of $\mathbb{R}$.

2.5) Prove that if a subset S of $\mathbb{R}$ is closed then $\boldsymbol{S}$ **contains** all its **accumulation points.**

[4,6,5,4,6]=[25]

QUESTION 3

3.1) Consider the sequence $\left\{\frac{3n+1}{n+1}\right\}_{n=1}^{\infty}$. Use the (ε, N)**-definition** to prove that $\lim\limits_{n\to\infty} \frac{3n+1}{n+1} = 3$.

3.2) Prove that a **convergent** sequence $\{a_n\}$ is a **Cauchy sequence.**

3.3) Determine whether $\{\sqrt{n^2+n} - \sqrt{n^2-1}\}$ is a **Cauchy** sequence.

3.4) Let $f: (0, 7) \to \mathbb{R}$ be defined by the function $f(x) = x^2 + 2x - 5$. Show that f is **uniformly continuous** on the interval $(0, 7)$.

[4, 5, 4, 6] = [19]

QUESTION 4

4.1) Let $f : D \to \mathbb{R}$ be **differentiable** at x_0. **Prove** that f is **continuous at** x_0.

4.2) Is it true since $f(x) = |x+1|$ is **continuous** at $x = -1$ it implies that f is **differentiable** at $x = -1$? If the statement is **false, explain.**

4.3) Suppose f is **continuous** on $[a, b]$, **differentiable** on (a, b).
Prove if $f'(x) \neq 0, \forall x \in (a, b)$, then f is **one-to-one.**

4.4) Use **4.3)** to show that $f(x) = e^{x^3+x}$ is **one-to-one.**

[5,2,5,3] =[15]

QUESTION 5

5.1) Show that $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = \mathbf{1}.$

5.2) Using the convergent geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, |x| \leq \mathbf{1}$ **to compute** the **Maclaurin series** for $f(x) = \frac{1}{1+2x}$.

5.3) Use an **appropriate test** to show that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent.

5.4) Determine whether the **relation** ~ on $\mathbb{Z}$ defined by $\boldsymbol{a \sim b}$ **iff** $2|\boldsymbol{ab}$ is an **equivalence relation**?

On equivalence relation

[5,3,3,3]=[14]

END