A COMPARATIVE STUDY OF ONE-STEP METHODS FOR NUMERICAL SOLUTION OF INITIAL-VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS

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A COMPARATIVE STUDY OF ONE-STEP METHODS FOR NUMERICAL SOLUTION OF INITIAL-VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS

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DECLARATION

I, Phumezile Kama do hereby declare that the work contained in this dissertation is entirely my own work with the exception of such quotations and references which have been attributed to their authors or sources and that this dissertation has not been submitted previously for any degree at any University.

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DEDICATION

This dissertation is dedicated to my Grandmother Dadi, my parents Ringi and Nomsokolo, my brothers Stalman and Mzwandile, my sister Ntombekhaya and my lovely daughter Mandisa.

ABSTRACT

The ultimate aim of the field of Numerical Analysis is to provide a reliable and convenient methods for obtaining useful solutions to mathematical problems and for extracting useful information from available solutions which are not expressed in proper form.

It is a common norm that analytic solutions, when available may be very precise in themselves but may be, of course, of unacceptable form because of the fact that they are not amenable to direct interpretation in numerical terms. In which case numerical analyst may attempt to devise a method of effecting that interpretation in a satisfactory way, or he may prefer to base his analysis instead upon the original formulation.In this dissertation, we shall carry out a brief but comprehensive study of some of the existing one-step methods for numerical solution of ordinary differential equations. We also proposed a relatively new method. Numerical results were obtained for the proposed methods and the results compared with the existing results.

SUMMARY

The outline of this dissertation is divided into four stages

Chapter one is introductory and involves a careful study of existing literature to identify some existing methods and some existing physical problems that may be of great interest during the course of this research. At this stage, various one-step methods will be studied.

Chapter two contains brief but comprehensive derivation of some one-step methods for numerical solution of ordinary differential equations. At this stage some of the existing one-step methods are carefully analysed. The comparative study of their Convergence, Stability, and Consistency properties is carried out.

In chapter three computer implementation of some sample test problems are carried out. Programs are written in TRUE BASIC. The results obtained for various methods are compared.

In chapter four we present general conclusions and comments. We also discuss problems for which the one-step method fails and possible areas of extension.

CHAPTER ONE

INTRODUCTION

1.1 ORDINARY DIFFERENTIAL EQUATIONS

Ordinary differential equations (ODE) arise in a variety of subject areas which include not only physical sciences but also such diverse fields as economics, medicine, psychology and operations research, management and social sciences. We will find applications of differential equations when the situation deals with rates of change of one variable with respect to another.

On a more practical level, it could be claimed that the spread of modern industrial civilization is partly a result of man's ability to solve differential equations which govern so many of our industrial processes. Many mathematical techniques now play an important role in planning, managerial decision-making, and economics which probably been longest quantified of the social sciences.

For example, let us consider the logistic population growth model. The basic assumption of the pure exponential model is that the rate of increase of population is proportional to the size of the population. The model assumes that sufficient resources are available to sustain any level of population so that there is no interference between individuals in the population. These assumptions are not realistic. Every species of organism inhabits some restricted environment, with a finite amount of space and a limited supply of resources. As the population gets closer to the 'carrying capacity', its rate of growth must slow down. Any realistic model of population dynamics should reflect this feature.The mathematical model should then assert that the rate of population is in fact a function of the population; mathematically, the situation is represented by a simple differential equation of the form:

$$\frac{dP}{dt} = f(P) \tag{1.1}$$

Where $P = P(t)$ is the population at time t and f is some function of population size P. This model can also be employed to describe a population in which both births and deaths occur. Assuming that the birth rate b and the death rate d are positive constants, independent of time, size of population, and age of individual. The model is the differential equation

$$\frac{dP}{dt} = (b - d)P. \tag{1.2}$$

Setting $a = b - d$, we see that the same equation

$$\frac{dP}{dt} = aP \tag{1.3}$$

describes all three situations. (c.f. Olinich, M. 1978)

To illustrate the use of differential equations in the social and management sciences, we consider the mathematical problem of describing the variation in sales of a products with different amounts of advertising expenditures. Let us assume that sales is a function of advertising only and that in the absence of advertising there are no sales. Furthermore, assume that advertising merely obtain new customers from a finite population, so that increased customer usage of the product is neglected. Denote the total sales by S, the advertising expenditure by A, and the maximum market potential by b. Since sales are postulated to be increasing function of advertising, we know $dS/dA > 0$. Initially, $S = 0$ when $A = 0$. The first advertising expenditure are most effective because the total market b is available. As sales approach the maximum market potential, $S \to b$, the rate of change in sales decreases; that is, $d^2S/dA^2 < 0$. Thus, dS/dA is a function of the untapped market potential, $b - S$:

$$\frac{dS}{dA} = f(b - S). \tag{1.4}$$

The simplest function f demonstrating the behaviour that (1.4) is positive and decreases to zero as $S \to b$ is

$$f(b - S) = \alpha(b - S), \tag{1.5}$$

where $\alpha(> 0)$ is constant. Therefore, under our assumption, the simplest mathematical model describing the response of sales to advertising is given by the first-order ordinary differential equation

$$\frac{dS}{dA} = \alpha(b - S). \tag{1.6}$$

with the initial condition $S = 0$ when $A = 0$. (c.f. Hayes, P. 1975)

Firstly, we begin with some common terminology. If an equation involves the derivative of one variable with respect to another, then the former is called a dependent variable and the latter is an independent variable. Thus,in the equation

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = 0 \tag{1.7}$$

t is the independent variable and x is the dependent variable. We refer to a and k as parameters or coefficients.

A differential equation involving ordinary derivatives with respect to a single independent variable is called an ordinary differential equation. A differential equation involving partial derivatives with respect to one or more independent variables is a partial differential equation. Ordinary differential equations are classified according to their order. The order of a differential equation is the order of the highest derivative present in the differential equation and its degree is the degree of the derivative of the highest order after the equation has been rationalized.

It will be useful to classify ordinary differential equations as either linear or nonlinear. A linear differential equation is any equation that can be written in the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x) \tag{1.8}$$

Where $a_n(x), a_{n-1}(x), \cdots a_0$, and $F(x)$ depend only on the independent variable x, not y. If an ordinary differential equation is not linear, then we call it nonlinear. If $F(x)$ is identically zero, the equation is said to be homogeneous; otherwise it is called nonhomogeneous.

Throughout this dissertation we shall use prime ($'$) to denote the derivatives.

Definition 1 [Solution of differential equation].

Any nth order ordinary differential equation can be expressed in the general form

$$F(x, y, y', \cdots, y^n) = 0 \tag{1.9}$$

where F is a function of the independent variable x, the dependent variable y, and the derivatives of y up to order n. We assume that x lies in some interval I. This equation can be expressed in a normal form;

$$y^{(n)} = f(x, y, y', \cdots, y^{(n-1)}) \tag{1.10}$$

A function $\phi(x)$ that when substituted for y in (1.9) or (1.10) is said to be a solution of a differential equation if together with its derivatives satisfies the equation for all x in the interval I.

A differential equation of order n will generally have a solution involving n arbitrary constants. This solution is called the general solution. If we assign a definite value to the constants, then solution so obtained is called a particular solution.

1.2 INITIAL VALUE PROBLEM FOR SYSTEM OF FIRST ORDER ODES

A first order differential equation may possess a unique solution, no solution at all or have an infinite number of solutions. In order to determine the arbitrary constants in the general solution the n conditions are prescribed at one point, these are called initial conditions. The differential equation together with the initial condition is called the initial value problem (IVP).If the n conditions are prescribed at more than one points, these are called boundary conditions. The differential equation together with boundary conditions is known as the boundary value problem (BVP).

Definition 2 [initial value problem]

Let $f(x, y)$ be a continuous function of x and y. The initial value problem is to solve

$$y'(x) = f(x, y(x)) \quad \text{with} \quad y(a) = y_0 \quad x \in [a, b], \quad y \in \Re \tag{1.11}$$

The nth order initial value problem can be expressed as

$$y^{(n)} = f(x, y, y', \cdots, y^{(n-1)}) \quad y(x_0) = y_0 \tag{1.12}$$

where x_0 is a specified value. The first order initial value problem (1.12) is equivalent to the following system of n first order equations

$$\begin{aligned} y_1 &= y_0 \\ y_2 &= y_0' \end{aligned}$$

$$y_n = y_0^{(n-1)} \tag{1.13}$$

Then (1.12) becomes

$$y_n' = f(x, y_1, y_2, \cdots, y_n) \tag{1.14}$$

Differentiating (1.13), we obtain

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \end{aligned}$$

$$\begin{aligned} y_{n-1}' &= y_n \\ y_n' &= f(x, y_1, y_2, \cdots, y_n) \end{aligned}$$

In vector notation it can be written as

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}), \qquad \mathbf{y}(x_0) = \mathbf{y}_0 0$$

where

$$\begin{aligned} \mathbf{y} &= [y_1, y_2, \cdots, y_n]^T \\ \mathbf{f}(x, \mathbf{y}) &= [y_2, y_3, \cdots, f(x, y_1, y_2, \cdots, y_n)]^T \\ \mathbf{y}_0 &= \left[y_0, y_1, \cdots, y_0^{(n-1)}\right]^T \end{aligned}$$

We shall, therefore, be concerned with the method for finding out numerical approximation to the solution of the equation (1.11). However, before attempting to approximate the solution numerically, we must ask if the problem has any solution. This can be answered in the case of initial value problems for ordinary differential equations by the following theorem.

Theorem 1 [Henrici (1962)]

let f be defined and continuous on the strip $S = \{(x,y) \mid a \leq x \leq b, \quad y \in (-\infty,\infty)\}$ $\quad a,b$ finite. Further, let there be a constant L such that

$$| f(x,y) - f(x,y^*) | \leq L \mid y - y^* | \tag{1.15}$$

for all $x \in [a,b]$ and all $y, y^* \in (-\infty,\infty)$ (Lipschitz condition). Then for every $x_0 \in [a,b]$ and every $y_0 \in (-\infty,\infty)$ there exists exactly one function $y(x)$ such that

(a) $y(x)$ is continuous and continuously differentiable for $x \in [a,b]$;

(b) $y'(x) = f(x,y(x))$ for $x \in [a,b]$

(c) $y(x_0) = y_0$.

In particular, if $f(x,y)$ possesses a derivative with respect to y, then by the Mean Value Theorem it follows that the Lipschitz condition is satisfied if the partial derivatives

$$f(x,y) - f(x,y^*) = \frac{\partial f(x,\bar{y})}{\partial y}(y - y^*) \tag{1.16}$$

exist on the strip S and are continuous and bounded there. $\bar{y}$ is a point in the interior of the interval

It follows that (1.16) can be satisfied if we choose

$$L = \sup_{(x,y) \in S} \frac{\partial f(x,\bar{y})}{\partial y} \tag{1.17}$$

We refer to the homogeneous solution of the initial value problem as the transient solution and the particular solution is commonly referred to as the steady state solution.

Theorem 2 [Mathews (1987)]

The initial value problem (1.11) is well-posed if $f(x, y)$ is differentiable with respect to y and there exists a constant L so that $| f_y(x, y) | \leq L$ throughout S.

Definition 3 [Conditioning]

Consider the initial value problem (1.11) over the interval $[a, b]$. If $f_y(x, y) \leq 0$ for $a < x < b$, then the initial value problem is said to be well-conditioned over the interval $[a, b]$. If $f_y(x, y) > 0$ for $a < x < b$, then the initial value problem is ill-conditioned.

1.3 MOTIVATION AND SOME FUNDAMENTAL CONCEPTS

The popularity and the behaviour of one-step methods especially in solving some special initial value problem coupled with the celebrated works of Fatunla (1978) and Ibijola (1998) form the source of motivation for this dissertation. In this dissertation we shall restrict ourselves to such methods which are suitable for equations of the form (1.11). We are going to make use of the notation $x_n = x_0 + nh$, y_n is the value of y obtained using our numerical method. The value y_{n+1} may then appear either as a function of just one y-value y_n, or as a function of several values $y_n, y_{n-1}, \cdots, y_{n-p}$.In the first case we have a one-step method, while in the second case a multistep method.

We shall move forward to discuss some of the concepts relating to ordinary differential equations. We shall be concerned with methods for numerical approximation to the solution of the initial value problems (1.11). By a one-step method we mean a method whereby y_n is the only input data to the step in which y_{n+1} is to be computed, $n = 1, 2, ...$ At each step there is a small disturbance - truncation error and/or rounding error - which produces a similar transition to "another track" in the family of solution curves.

Euler proposed the simplest and most analysed numerical integration method. It is a one-step method for initial value problem (1.11) that can be described as follows:

$$y_{n+1} = y_n + h\phi(x_n, y_n; h) \tag{1.18}$$

where $\phi(x, y; h)$ is the increment function and h is the step size adopted in the subinterval $[x_n, x_{n+1}]$

Example 1

Let the given function f be $f(x, y) = y$. The differential equation

$$y' = y$$

is satisfied by the Euler's method

$$y_{n+1} = y_n + hy_n \qquad y_0 = 1$$

The solution is computed first with $h = 0.2$ and then with $h = 0.1$ and the results are shown in the table below

Table 1

x_t	Theoretical solution $y(x_t)$	Numerical solution Example 1		Error	
		$h = 0.2$	$h = 0.1$	$h = 0.2$	$h = 0.1$
0.0	1.0000000	1.00000	1.0000000	0.0000000	0.000000
0.1	1.1051709	—-	1.1000000	—-	5.170918e-3
0.2	1.2214028	1.20000	1.2100000	2.140276e-2	1.140276e-2
0.3	1.3498588	—-	1.3310000	—-	1.885881e-2
0.4	1.4918247	1.44000	1.4641000	5.182247e-2	2.772469e-2
0.5	1.6487213	—-	1.6105100	—-	3.821127e-2
0.6	1.8221188	1.72800	1.7715610	0.0941188	0.0505578
0.7	2.0137527	—-	1.9487171	—-	6.503561e-2
0.8	2.2255409	2.07360	2.1435888	0.15194093	8.195212e-2
0.9	2.4596031	—-	2.3579477	—-	0.10165542
1.0	2.7182818	2.48832	2.5937425	0.22996183	0.12453937

We observe that the Euler's method converges much slower to the theoretical solution.

The best known one-step methods are the Runge-Kutta methods. They are fairly simple to program and their truncation error can be controlled in a more straightforward manner. The major advantage of the Runge-Kutta methods is that they use many more evaluations of the derivative $f(x, y)$ to obtain the same accuracy, as compared to multistep method.

Some of the popular one-step method is based on Taylor series expansion. If $y(x)$ is n+1 times continuously differentiable, where $y(x)$ is the solution of initial value problem (1.11), expanding $y(x)$ about x_0 using Taylor's theorem, we obtain

$$y(x) = y(x_0) + hy'(x_0) + \cdots + \frac{h^n}{n!}y^n(x_0) + \frac{h^{n+1}}{(n+1)!}(\xi_0) \tag{1.19}$$

for some $x_0 \leq \xi_0 \leq x$. By dropping the remainder term, we have an approximation for $y(x)$, provided we can calculate $y''(x_0), \cdots, y^n(x_0)$. We differentiate $y' = f(x, y(x))$ to obtain

$$\begin{aligned} y''(x) &= f_x(x, y(x)) + f_y(x, y(x))y' \\ y'' &= f_x + f_y f \end{aligned}$$

Example 2

Consider the problem

$$y' = -y^2 \qquad y(0) = 1 \qquad h = 0.1$$

with solution $y(x) = \frac{1}{1+x}$, then $y'' = -2yy' = -2y^3$

and (1.19) with $n = 1$ yields

$$y(x) = y_0 + h{y_0}^2 + \frac{h^2}{2}y_0^3 + \frac{h^3}{6}y^{(3)}(\xi_0), \qquad x_0 \leq \xi_0 \leq x$$

We drop the remainder to obtain an approximation for $y(x)$. The numerical method is

$$y_{n+1} = y_n + h{y_n}^2 + \frac{h^2}{2}y_n^3, \quad n \geq 0 \tag{1.20}$$

Numerical results for example 2 are shown in the table below

Table 2

x_t	Theoretical solution $y(x_t)$	Numerical solution Example 2	Error
0.0	1.00000000	1.00000000	0.000000
0.1	0.90909091	0.91000000	9.090909e-4
0.2	0.83333333	0.83472571	1.392377e-3
0.3	0.76923077	0.77086510	1.634333e-3
0.4	0.71428571	0.71602254	1.736822e-3
0.5	0.66666667	0.66842467	1.758006e-3
0.6	0.62500000	0.62673198	1.731983e-3
0.7	0.58823529	0.58991444	1.679151e-3
0.8	0.55555556	0.55716744	1.611188e-3
0.9	0.52631579	0.52785353	1.537737e-3
1.0	0.50000000	0.50146135	1.461347e-3

These numerical results show agreement with the theoretical.

Runge-Kutta and Euler type methods are called one-step methods because they use only the information from the last step computed. In this they have the ability to perform the next step with a different step size and are ideal for beginning the solution where only the initial conditions are available. The weakness of Euler's method is that the step length must be chosen quite small in order to attain acceptable accuracy.

Definition 4 [Consistency]

The integration formula (1.18) is said to be consistent with the initial value problem (1.11) provided the increment function $\phi(x, y; h)$ satisfies the following relationship:

$$\phi(x, y; 0) = f(x, y).$$

The consistency of a one-step numerical integrator ensures that the scheme is at least of order one. Basically, consistency of a formula ensures that the method approximate the ordinary differential equation in some sense.

Definition 5 [Local Truncation Error]

The local truncation error (l.t.e) of the one-step scheme (1.18) is given by

$$T_{n+1} = y(x_{n+1}) - y(x_n) - h\phi(x_n, y(x_n); h),$$

where $y(x)$ is the theoretical solution to (1.11). The local truncation error is simply the amount by which the theoretical solution of (1.11) fails to satisfy the first order difference equation.

Definition 6 [Global error]

The global error e_{n+1} of (1.18) is the difference between the theoretical solution $y(x_{n+1})$ and the numerical solution y_{n+1};

$$e_{n+1} = y_{n+1} - y(x_{n+1})$$

Definition 7 [Convergence]

The one-step scheme (1.18) is considered convergent if, for any arbitrary initial solution vector y_0 and an arbitrary point $x \in [a, b]$, the global error fulfills the following relationship:

$$\lim_{n \to 0} \max e_n \to 0,$$

provided x is always a meshpoint.

Definition 8 [Stability]

The integration scheme (1.18) is stable, if for any initial error e_0, there exist constants K and $h_0 > 0$ such that, when (1.18) is applied to the initial value problem (1.11) with meshsize $h \in (0, h_0]$, the ultimate error e_n satisfies the following inequality:

$$e_n \leq K e_0, \qquad K < \infty.$$

1.4 EXISTING METHODS

Several authors have developed many numerical integrating schemes to generate the numerical solutions to problems of the form (1.11). Many of the these schemes include those developed by Gautschi (1961), Fatunla (1976, 1978a, 1978), Lambert (1973), Amdursky and Viz (1974), Bulirsh and Stoer (1960), Gragg (1965), and Krogh (1979). Fatunla (1988 page 21) pointed out that existing numerical integration schemes can be classified int three distinct categories:

(a) The first category comprises those schemes in which the approximate solutions are given by a linear combination of independent functions.

Examples are

(i) the Taylor series approach,

(ii) the Lie series approach, and

(iii) the Picard iteration scheme.

The three devices generate series of solution whose convergence cannot be guaranteed, and even when the series do converge, the rate of convergence may be rather low.

(b) The second class of numerical methods for (1.11) is the expansion methods whereby the theoretical solution $y(x)$ is approximated by the first few terms of an expansion in orthogonal functions

$$y_n(x) = \sum_{i=1}^{n} c_i H_i(x), \tag{1.21}$$

where the coefficients c_i satisfy the so called normal equations

$$\sum_{i=1}^{n}(H_i, H_j) = (y, H_j) \quad (j = 1, 2, ..., n).$$

When $H_1, H_2, ..., H_n$ are linearly independent, the approximation has a unique solution. The solution is characterised by the orthogonality property that $y_n - y$ is orthogonal to all H_i, $\quad (i = 1, 2, ..., n)$. An important special case is when $H_1, H_2, ..., H_n$ form an orthogonal system; then the coefficients are computed more simply by the formula

$$c_i = \frac{(y, H_i)}{(H_i, H_i)}.$$

The coefficients c_i are called the orthogonal coefficients.

(c) Finally, are the discrete variable methods whereby numerical approximations are obtained at some specified points in the integration interval. Some of these methods can be described as a more generally applicable version of Taylor series. This approach is more universally applicable, since in general, it is more amenable to computer implementation.

We shall now examine, without much details, some of the innovations which have been built into various ordinary differential equations to improve their efficiency, reliability and robustness.

The first order Backward Difference formula is the simplest method to solve the differential algebraic equation

$$F(x, y, y') = 0 \tag{1.22}$$

leading to the nonlinear equation

$$F(x_{n+1}, y_{n+1}, \frac{(y_{n+1} - y_n)}{(x_{n+1} - x_n)}) = 0 \tag{1.23}$$

The backward difference formula, Runge-Kutta methods, and Extrapolation methods are considered generalizations of this idea.

The approach suggested by Shampine (1987) is practical, cheap, efficient, reliable, and applicable to both the Adams and Backward Differentiation formula. It also eliminates the mesh distortion problem associated with Sedgwick's (1973) method of taking the smallest permissible machine unit as the initial stepsize and then building up both the order and stepsize very rapidly for the Adams method. The original predictor-corrector implementation of the Adams method in Shampine and Gordon (1975) is to generate two numerical solutions y_1 and y_1^* at $x_1 = x_0 + h$, adopting the explicit Euler and the trapezoidal scheme, respectively, with h just small enough to satisfy the condition

$$y_1^* - y_1 = \frac{1}{2}hf_1 - f_0 \le TOL, \tag{1.24}$$

where TOL is the error tolerance.

In order to guarantee the credibility of the initial stepsize, Shampine (1987) advocated that the trial initial stepsize, in addition to satisfying (1.24), should also satisfy the more stringent constraint

$$\frac{1}{2}h[f_1 + f_0] \le TOL, \tag{1.25}$$

and that a trial stepsize

$$h = \frac{1}{5}TOL^* f_0^{-1}, \tag{1.26}$$

be used to generate y_1 and y_1^* provided that $f_0 \neq 0$.

With an initial stepsize which satisfies Equations (1.24) and (1.25), an optimal stepsize

$$h_{opt} = (TOL^* y_1^* - y_1^{-1})^* h \tag{1.27}$$

is the largest stepsize for which all $h \le h_{opt}$ satisfy (1.24) provided the asymptotic behaviour of the formula is evident.

Zudanaisky (1976) proposed an ingenious method to generate global errors by formulating a Neighboring Problem

$$z' = f(x, y) + d(x), \qquad z(x_0) = y_0 \tag{1.28}$$

with

$$d(x) = P_k'(x) - f(x, P_k(x)) \tag{1.29}$$

for the IVP (1.11). Equation (1.28) is an artificial problem in that its exact solution $z(x) = P_k(x)$ is a piecewise polynomial of degree $\leq k$ interpolating the numerical solution y_n to (1.11). The corresponding numerical solution $\{z_n\}$ can be generated by the same numerical method used to generate $\{y_n\}$.

The global error for the solution to the IVP (1.11)

$$E_n = y_n - y(x_n) \tag{1.30}$$

is estimated by the global error for the neighboring problem (1.28)

$$W_n = z_n - P_k(x_n). \tag{1.31}$$

The three basic steps in the defect correction approach for any given differential system and a particular numerical integrator are:

(a) Obtain the defect of a numerical solution as a measure of how well the given problem has been solved,

(b) Use this defect in a simplified version of the problem to obtain the approximate correction quantity. This step normally involves the computation of an approximate Jacobian for the continuous error equation

$$e'(x) = f_y(x, y(x))e(x) + d(x), \tag{1.32}$$

with the global error given as

$$E(x_n) = h^p e(x_n). \tag{1.32}$$

(c) Apply the correction to the approximate solution to give improved values.

(c.f. Fatunla 1988)

CHAPTER TWO

2.1 A ONE-STEP METHOD BASED ON POLYNOMIAL INTERPOLANT

We shall consider the initial value problem of the form (1.11), namely

$$y'(x) = f(x, y(x)) \quad \text{with} \quad y(a) = y_0 \quad x \in [a, b], \quad y \in \Re.$$

We present the theoretical solution $y(x)$ to the initial value problem (1.11) in the interval $[x_t, x_{t+1}]$ by the polynomial

$$F(x) = a_0 + a_1 x + a_2 x^2 \tag{2.1}$$

where a_0, a_1 and a_2 are undetermined coefficients.

If y_t is a numerical estimate to the theoretical solution $y(x_t)$ and $f_t = f(x_t, y_t)$ with mesh points defined by $x_t = a + th, \quad t = 0, 1, 2, \ldots$ we can impose the following assumptions on the interpolant (2.1): At the points $x = x_t$ and $x = x_{t+1}$ the interpolant function (2.1) can be expressed as

$$F(x_t) = a_0 + a_1 x_t + a_2 x_t^2 \tag{2.2}$$

and

$$F(x_{t+1}) = a_0 + a_1 x_{t+1} + a_2 x_{t+1}^2 \tag{2.3}$$

To determine the undetermined coefficients a_1 and a_2 we rely on some specific assumptions. We also note that a_0 vanishes naturally. We also assume that the interpolating function (2.2) coincides y_t and (2.3) coincides with y_{t+1}. That is,

$$F(x_t) = a_0 + a_1 x_t + a_2 x_t^2 = y_t \tag{2.4}$$

$$F(x_{t+1}) = a_0 + a_1 x_{t+1} + a_2 x_{t+1}^2 = y_{t+1} \tag{2.5}$$

We also require that the first and second derivatives of the interpolant function respectively coincide with the differential equation as well as its first derivative with respect to x at $x = x_t$. Let $f^{(i)}$ denote the ith total derivative of $f(x, y)$ with respect to x such that

$$F^{(1)}(x_t) = f(x_t, y_t) = f_t \tag{2.6}$$

and

$$F^{(2)}(x_t) = f^{(1)}(x_t, y_t) = f_t^{(1)}. \tag{2.7}$$

Differentiating (2.2) with respect to x, we obtain

$$F^{(1)}(x_t) = a_1 + 2a_2 x_t = f_t \tag{2.8}$$

and

$$F^{(2)}(x_t) = 2a_2 = f_t^{(1)}. \tag{2.9}$$

Solving for a_2 from equation (2.9), gives

$$a_2 = \frac{1}{2} f_t^{(1)} \tag{2.10}$$

Substituting (2.10) into (2.8) and rearranging terms yields

$$a_1 = f_t - x_t f_t^{(1)} \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.2), we have

$$\begin{aligned} F(x_t) &= a_0 + x_t(f_t - x_t f_t^{(1)}) + \frac{1}{2} x_t^2 f_t^{(1)} \\ &= a_0 + x_t f_t - \frac{1}{2} x_t^2 f_t^{(1)} \end{aligned}$$

which can be written in the form

$$y_t = a_0 + x_t f_t - \frac{1}{2} x_t^2 f_t^{(1)} \tag{2.12}$$

and substituting (2.10) and (2.11) into (2.3) yields

$$\begin{aligned} F(x_{t+1}) &= a_0 + x_{t+1}(f_t - x_t f_t^{(1)}) - \frac{1}{2} x_{t+1}^2 f_t^{(1)} \\ &= a_0 + x_{t+1} f_t + (\frac{1}{2} x_{t+1}^2 - x_{t+1} x_t) f_t^{(1)} \end{aligned}$$

which by (2.5) can be written in the form

$$y_{t+1} = a_0 + x_{t+1} f_t + (\frac{1}{2} x_{t+1}^2 - x_{t+1} x_t) f_t^{(1)} \tag{2.13}$$

Subtracting equation (2.12) from (2.13), gives

$$\begin{aligned} y_{t+1} - y_t &= (x_{t+1} - x_t) f_t + [(\frac{1}{2} x_{t+1}^2 - x_{t+1} x_t) + \frac{1}{2} x_t^2] f_t^{(1)} \\ &= (x_{t+1} - x_t) f_t + \frac{1}{2}(x_{t+1}^2 - 2x_{t+1} x_t + x_t^2) f_t^{(1)} \\ &= (x_{t+1} - x_t) f_t + \frac{1}{2}(x_{t+1} - x_t)^2 f_t^{(1)} \\ &= h f_t + \frac{1}{2} h^2 f_t^{(1)} \end{aligned}$$

where $h = x_{t+1} - x_t$ after using $x_t = a + th$ and $x_{t+1} = a + (t+1)h$.

Hence the above numerical scheme can be written in the form

$$y_{t+1} = y_t + h f_t + \frac{1}{2} h^2 f_t^{(1)} \tag{2.14}$$

2.2 PROPOSED ONE-STEP METHOD BASED ON A PERTURBED INTERPOLANT

We shall proceed further to introduce an exponential term $e^{\lambda x^2}$ to equation (2.1) of the above section. The approach here is based on a representation of the solution by a combination of a polynomial and exponential function. In this section we wish to investigate the behaviour of the proposed function based on assumptions made in section 2.1. That is,

$$F(x) = a_0 + a_1 x + a_2 x^2 + a_3 e^{\lambda x^2} \tag{2.15}$$

The behaviour of this function depends on the sign of λ. If λ is positive, $F(x)$ is a steadily increasing function of x. If λ is zero, $F(x)$ reduces to equation (2.1) for all x. If λ is negative, then $F(x)$ is a steadily decreasing function of x that approaches equation (2.1) as x grows large.

We are interested in the numerical method arising from (2.15). At the points $x = x_t$ and $x = x_{t+1}$, we have

$$F(x_t) = a_0 + a_1 x_t + a_2 x_t^2 + a_3 e^{\lambda x_t^2} = y_t \tag{2.16}$$

and

$$F(x_{t+1}) = a_0 + a_1 x_{t+1} + a_2 x_{t+1}^2 + a_3 e^{\lambda x_{t+1}^2} = y_{t+1} \tag{2.17}$$

To determine a_1, a_2, and a_3 we require the first, second and the third derivatives of $F(x)$ coinciding with the differential equation as well as its first derivative with respect to x at $x = x_t$.

$$F^{(1)}(x_t) = a_1 + 2a_2 x_t + a_3(2\lambda x_t e^{\lambda x_t^2}) = f_t \tag{2.18}$$

$$F^{(2)}(x_t) = 2a_2 + a_3(2\lambda e^{\lambda x_t^2} + 4\lambda^2 x_t^2 e^{\lambda x_t^2}) = f_t^{(1)} \tag{2.19}$$

$$F^{(3)}(x_t) = a_3(12\lambda^2 x_t e^{\lambda x_t^2} + 8\lambda^3 x_t^3 e^{\lambda x_t^2}) = f_t^{(2)} \tag{2.20}$$

After some simple manipulations, (2.20) reduces to

$$a_3 = \frac{1}{(12\lambda^2 x_t + 8\lambda^3 x_t^3)e^{\lambda x_t^2}} f_t^{(2)} \tag{2.21}$$

Substituting (2.21) into (2.18) and solving for a_2, gives

$$a_2 = \frac{1}{2} f_t^{(1)} - \frac{(\lambda + 2\lambda^2 x_t^2)}{12\lambda^2 x_t + 8\lambda^3 x_t^3} f_t^{(2)} \tag{2.22}$$

It follows from (2.21) and (2.22) that

$$\begin{aligned} a_1 &= f_t - 2a_2 x_t - 2\lambda a_3 x_t e^{\lambda x_t^2} \\ &= f_t - x_t f_t^{(1)} + \frac{2x_t(\lambda + 2\lambda^2 x_t^2)}{12\lambda^2 x_t + 8\lambda^3 x_t^3} f_t^{(2)} - \frac{2\lambda x_t}{12\lambda^2 x_t + 8\lambda^3 x_t^3} f_t^{(2)} \end{aligned}$$

so that

$$a_1 = f_t - x_t f_t^{(1)} + \frac{4\lambda^2 x_t^3}{12\lambda^2 x_t + 8\lambda^3 x_t^3} f_t^{(2)} \tag{2.23}$$

The insertion of (2.21), (2.22) and (2.23) in (2.16) yields

$$\begin{aligned} F(x_t) &= a_0 + x_t f_t - x_t^2 f_t^{(1)} + \frac{4\lambda^2 x_t^4}{12\lambda^2 x_t + 8\lambda^3 x_t^3} f_t^{(2)} + \frac{1}{2} x_t^2 f_t^{(1)} \\ &\quad - \frac{x_t^2(\lambda + 2\lambda^2 x_t^2)}{12\lambda^2 x_t + 8\lambda^3 x_t^3} f_t^{(2)} + \frac{1}{12\lambda^2 x_t + 8\lambda^3 x_t^3} f_t^{(2)} \end{aligned}$$

which can be written as

$$y_t = a_0 + x_t f_t - \frac{1}{2} x_t^2 f_t^{(1)} + \frac{2\lambda^2 x_t^4 - \lambda x_t^2 + 1}{12\lambda^2 x_t + 8\lambda^3 x_t^3} f_t^{(2)} \tag{2.24}$$

In a similar manner substituting (2.21), (2.22) and (2.23) into (2.17), we have

$$\begin{aligned} F(x_{t+1}) &= a_0 + x_{t+1} f_t - x_{t+1} x_t f_t^{(1)} + \frac{4\lambda^2 x_{t+1} x_t^3}{12\lambda^2 x_t + 8\lambda^3 x_t^3} f_t^{(2)} + \frac{1}{2} x_{t+1}^2 f_t^{(1)} \\ &\quad - \frac{x_{t+1}^2(\lambda + 2\lambda^2 x_t^2)}{12\lambda^2 x_t + 8\lambda^3 x_t^3} f_t^{(2)} + \frac{e^{\lambda(x_{t+1}^2 - x_t^2)}}{12\lambda^2 x_t + 8\lambda^3 x_t^3} f_t^{(2)} \end{aligned}$$

which can be written in form

$$y_{t+1} = a_0 + x_{t+1} f_t + (\frac{1}{2}x_{t+1}^2 - x_{t+1}x_t) f_t^{(1)} + \frac{4\lambda^2 x_{t+1} x_t^3 - \lambda x_{t+1}^2 - 2\lambda^2 x_{t+1}^2 x_t^2 + e^{-\lambda(x_{t+1}^2 - x_t^2)}}{12\lambda^2 x_t + 8\lambda^3 x_t^3} f_t^{(2)} \tag{2.25}$$

By subtracting (2.24) from (2.25), we obtain

$$\begin{aligned} y_{t+1} - y_t &= (x_{t+1} - x_t) f_t + (\frac{1}{2}x_{t+1}^2 - x_{t+1}x_t + \frac{1}{2}x_t^2) f_t^{(1)} + \\ &\quad [\frac{4\lambda^2 x_{t+1} x_t^3 - \lambda x_{t+1}^2 - 2\lambda^2 x_{t+1}^2 x_t^2 - 2\lambda x_t^4 + \lambda x_t^2 + e^{-\lambda(x_{t+1}^2 - x_t^2)} - 1}{12\lambda^2 x_t + 8\lambda^3 x_t^3}] f_t^{(2)} \\ &= (x_{t+1} - x_t) f_t + \frac{1}{2}(x_{t+1} - x_t)^2 f_t^{(1)} \\ &\quad + [\frac{2\lambda^2 x_t^2 (x_{t+1} - x_t)^2 + \lambda(x_{t+1} - x_t)}{12\lambda^2 x_t + 8\lambda^3 x_t^3}] f_t^{(2)} + [\frac{e^{-\lambda(x_{t+1}^2 - x_t^2)} - 1}{12\lambda^2 x_t + 8\lambda^3 x_t^3}] f_t^{(2)} \\ &= h f_t + \frac{1}{2}h^2 f_t^{(1)} + \\ &\quad [\frac{-2\lambda^2 h^2 (a^2 + 2aht + h^2 t^2) - \lambda h (2a + 2ht + h) + e^{-\lambda(2ah + 2h^2 t + h^2)} - 1}{12\lambda^2 (a + ht) + 8\lambda^3 (a + ht)^3}] f_t^{(2)} \end{aligned}$$

which can be expressed as

$$y_{t+1} = y_t + h f_t + \frac{1}{2}h^2 f_t^{(1)} + \left[\frac{-2\lambda^2 h^2 (a^2 + 2aht + h^2 t^2) - \lambda h (2a + 2ht + h) + e^{-\lambda(2ah + 2h^2 t + h^2)} - 1}{12\lambda^2 (a + ht) + 8\lambda^3 (a + ht)^3}\right] f_t^{(2)} \tag{2.26}$$

$$y_{t+1} = y_t + h\{f_t + \frac{1}{2}h f_t^{(1)} + [\frac{-2\lambda^2 h (a^2 + 2aht + h^2 t^2) - \lambda (2a + 2ht + h) + \frac{1}{h}(e^{-\lambda(2ah + 2h^2 t + h^2)} - 1)}{12\lambda^2 (a + ht) + 8\lambda^3 (a + ht)^3}] f_t^{(2)}\} \tag{2.27}$$

Equation (2.27) can be written in a compact form as

$$y_{t+1} = y_t + h(f_t + Mf_t^{(1)} + Nf_t^{(2)}) \tag{2.28}$$

where

$$M = \frac{h}{2}$$

and

$$N = \frac{-(2\lambda^2 a^2 h + 4a\lambda^2 h^2 t + 2\lambda^2 h^3 t^2) + 2a\lambda + 2\lambda h t + \lambda h) + \frac{1}{h}(e^{-\lambda(2ah+2h^2t+h^2)} - 1)}{12\lambda^2(a+ht) + 8\lambda^3(a+ht)^3}$$

Definition 9 [Henrici (1962)]

We define any method for solving a differential equation in which the approximation y_{t+1} to the solution at the point x_{t+1} can be calculated if only x_t, y_t and h are known as a ONE-STEP METHOD. It is a common practice to write the functional dependence y_{t+1} on the quantities x_t, y_t and h in the form

$$y_{t+1} = y_t + h\Phi(x_t, y_t; h) \tag{2.29}$$

where $\Phi(x_t, y_t; h)$ is the increment function.

In view of (2.28) and (2.29) it follows that

$$\Phi(x_t, y_t; h) = f_t + Mf_t^{(1)} + Nf_t^{(2)} \tag{2.30}$$

2.3 CONVERGENCE, CONSISTENCY AND STABILITY OF THE PROPOSED METHOD

2.3.1 CONVERGENCE

The concept of convergence plays an important part in the analysis of numerical methods. In this section we want to find the circumstances under which the numerical method (2.28) will be convergent.

We apply theorem 1 to investigate the convergence of (2.28)

Let the function $\Phi(x, y; h)$ be continuous in the region defined by $x \in [a, b]$, $y \in (-\infty, \infty)\}$ $0 \leq h \leq h_0$, where $h_0 > 0$, and let there exist a constant L such that

$$|\Phi(x, y^*; h) - \Phi(x, y; h)| \leq L \mid y^* - y \mid, \tag{2.31}$$

for all $(x, y; h)$ and$(x, y^*; h)$ in the region just defined. Then the relation

$$\Phi(x_t, y_t; 0) = f(x, y) \tag{2.32}$$

is a necessary and sufficient condition for the convergence of the method defined by the increment function Φ. With the increment function given by

$$\Phi(x_t, y_t; h) = f(x_t, y_t) + Mf^{(1)}(x_t, y_t) + Nf^{(2)}(x_t, y_t)$$

and

$$\Phi(x_t, y_t^*; h) = f(x_t, y_t^*) + Mf^{(1)}(x_t, y_t^*) + Nf^{(2)}(x_t, y_t^*)$$

we have

$$\begin{aligned}\Phi(x_t, y_t^*; h) - \Phi(x_t, y_t; h) &= f(x_t, y_t^*) - f(x_t, y_t) + M[f^{(1)}(x_t, y_t^*) - f^{(1)}(x_t, y_t)] \\ &\quad + N[f^{(2)}(x_t, y_t^*) - f^{(2)}(x_t, y_t)]\end{aligned}$$

Let $\bar{y}_t$ be a point in the interior of the interval whose endpoints are y and y^*, if we apply the Mean Value Theorem, we get

$$f(x_t, y_t^*) - f(x_t, y_t) = \frac{\partial f(x_t, \bar{y}_t)}{\partial y_t}(y_t^* - y_t), \tag{2.33}$$

$$f^{(1)}(x_t, y_t^*) - f^{(1)}(x_t, y_t) = \frac{\partial f^{(1)}(x_t, \bar{y}_t)}{\partial y_t}(y_t^* - y_t) \tag{2.34}$$

and

$$f^{(2)}(x_t, y_t^*) - f^{(2)}(x_t, y_t) = \frac{\partial f^{(2)}(x_t, \bar{y}_t)}{\partial y_t}(y_t^* - y_t). \tag{2.35}$$

By equation (1.17), we define

$$L_1 = \sup_{(x_t, \bar{y}_t) \in S} \frac{\partial f(x_t, \bar{y}_t)}{\partial y_t},$$

$$L_2 = \sup_{(x_t, \bar{y}_t) \in S} \frac{\partial f^{(1)}(x_t, \bar{y}_t)}{\partial y_t}$$

and

$$L_3 = \sup_{(x_t, \bar{y}_t) \in S} \frac{\partial f^{(2)}(x_t, \bar{y}_t)}{\partial y_t}.$$

Then

$$\mid \Phi(x_t, y_t^*; h) - \Phi(x_t, y_t; h) \mid \leq \mid L_1 + ML_2 + NL_3 \mid\mid y_t^* - y_t \mid = L \mid y_t^* - y_t \mid \tag{2.36}$$

2.3.2 CONSISTENCY

The concept of consistency of one-step method is very important in the sense that it controls the magnitude of the local truncation error, and is crucial to the convergence of one step-methods. We expect a convergent method to be at least consistent, however, consistency does not necessarily imply convergence.

By definition 4 a numerical method with an increment function $\Phi(x_t, y_t; h)$ is said to be consistent with the initial value problem (1.11), if

$$\Phi(x_t, y_t; 0) = f(x_t, y_t). \tag{2.37}$$

If we put $h = 0$ in (2.28) after substituting the values of M and N, we have equation (2.37) which confirms that the method is consistent.

2.3.3 STABILITY

In this section it will be shown how to investigate the stability of the numerical method (2.28). The definition of stability that we will use is related to our intuitive feeling that errors or other disturbances in the calculation of the solution to a numerical equation will not dominate the true solution. If the method introduces small errors into the problem, then instability of the problem will result in large errors. These errors should not be blamed on the method. In fact, the method may do quite well when applied to stable problems

In order to properly judge numerical methods, the stability of the method must not be confused with the behaviour of the problem. A simple definition of stable method would probably involve a comparison of the exact solution and computed solution. The difference between these solutions might be large only because the problem is unstable, not because the method is not ideal.

Instability exists in various forms but there are two basic types, namely, the inherent instability and the induced instability. Inherent instability is the property of the differential equation itself while induced instability is the characteristic of the numerical method.

Theorem

Let $y_t = y(x_t)$ and $z_t = z(x_t)$ denote two solutions of the differential equation (1.11) with the initial conditions specified as $y(x_0) = \eta$ and $z(x_0) = \eta^*$ respectively, such that $\mid \eta - \eta^* \mid < \epsilon, \quad \epsilon > 0$. If the two numerical estimates are generated by

$$y_{t+1} = y_t + h\Phi(x_t, y_t; h) \tag{2.38}$$

and

$$z_{t+1} = y_t + h\Phi(x_t, z_t; h). \tag{2.39}$$

The condition that

$$\mid y_{t+1} - z_{t+1} \mid \leq K \mid \eta - \eta^* \mid \tag{2.40}$$

is a necessary and sufficient condition that the numerical method (2.28) be stable and convergent.

Proof

We shall apply the statement of the above theorem to the numerical method (2.28). Let

$$y_{t+1} = y_t + h[f(x_t, y_t) + Mf^{(1)}(x_t, y_t) + Nf^{(2)}(x_t, y_t)] \tag{2.41}$$

and

$$z_{t+1} = z_t + h[f(x_t, z_t) + Mf^{(1)}(x_t, z_t) + Nf^{(2)}(x_t, z_t)]. \tag{2.42}$$

Then

$$\begin{aligned} y_{t+1} - z_{t+1} &= y_t - z_t + h[f(x_t, y_t) - f(x_t, z_t) + M\{f^{(1)}(x_t, y_t) - f^{(1)}(x_t, z_t)\} \\ &\quad + N\{f(2)(x_t, y_t) - f^{(2)}(x_t, z_t)\}] \end{aligned}$$

Applying the Mean Value Theorem as before, gives

$$| y_{t+1} - z_{t+1} | \leq | 1 + h(L_1 + ML_2 + NL_3 || y_t - z_t | \tag{2.43}$$

Hence given $\epsilon > 0$ and $y(x_0) = \eta \quad z(x_0) = \eta^*$, then

$$| y_{t+1} - z_{t+1} | \leq K | \eta - \eta^* | \tag{2.44}$$

where $K = 1 + h(L_1 + ML_2 + NL_3)$.

Since the solution is bounded, then it follows in the sense of (2.44) that it is stable and convergent.

CHAPTER THREE

3.1 APPLICATIONS AND NUMERICAL RESULTS

Test runs were made with the method for some initial value problems. The numerical computation was implemented and run on an IBM compatible machine. To obtain the numerical solution y_{t+1} at $x = x_{t+1}$, the function $f(x, y)$ and its higher derivatives are evaluated at $x = x_t$. The values thus obtained are used in equations (2.14) and (2.28) to generate y_{t+1}.

3.1.1 NUMERICAL RESULTS OF THE METHOD BASED ON POLYNOMIAL INTERPOLANT

In this section we are mainly concerned with the application of equation (2.14). The results obtained are compared with different step sizes.

Example 1

We first consider the scalar initial value problem

$$y' = y, \qquad y(0) = 1 \tag{3.1}$$

in the interval $0 \leq x \leq 1$ using a uniform step size $h = 0.2$ and $h = 0.1$. The solution is then compared to the exact solution $y(x) = e^x$. The details of the numerical results are given in Table 3.

Table 3

x_t	Theoretical solution $y(x_t)$	Numerical solution	Formula (2.14)		Error		
					$10^{-4}\times E$	$10^{-5}\times E$	$10^{-6}\times E$
		$h=0.2$	$h=0.1$	$h=0.05$	$h=0.2$	$h=0.1$	$h=0.05$
0.0	1.0000000	1.0000000	1.0000000	1.0000000	0.000000	0.000000	0.000000
0.1	1.1051709	—-	1.1050000	1.1051266	—-	0.170918	0.443556
0.2	1.2214028	1.2200000	1.2210250	1.2213047	0.140276	0.37775816	0.980390
0.3	1.3498588	—-	1.3492326	1.3496963	—-	0.626183	0.016252
0.4	1.4918247	1.4884000	1.4909021	1.4915852	0.342469	0.922647	0.023948
0.5	1.6487213	—-	1.6474468	1.6483904	—-	0.012745	0.033083
0.6	1.8221188	1.815848	1.8204287	1.8216801	0.627080	0.016901	0.043874
0.7	2.0137527	—-	2.0115737	2.0131870	—-	0.021790	0.056568
0.8	2.2255409	2.2153346	2.2227889	2.2248265	0.010206	0.027520	0.071447
0.9	2.4596031	—-	2.4561818	2.4587148	—-	0.034214	0.088829
1.0	2.7182818	2.7027082	2.7140808	2.7171911	0.015574	0.042009	0.001091

We notice that the errors are much smaller than those obtained in the Euler's method (see Table 1). This suggest that this method is more accurate than Euler's method.

Example 2

Compute a solution of

$$y' = xy^{\frac{1}{3}}, \qquad y(1) = 1 \tag{3.2}$$

with the step sizes $h = 0.1$, $h = 0.05$ and $h = 0.01$ and compare with the exact solution $y(x) = [\frac{x^2+2}{3}]^{\frac{3}{2}}$. The numerical results are illustrated in Table 4.

Table 4

x_t	Theoretical solution $y(x_t)$	Numerical solution	Formula (2.14)		Error $10^{-5} \times E$	$10^{-6} \times E$	$10^{-7} \times E$
		$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.1$	$h = 0.05$	$h = 0.01$
1.0	1.0000000	1.0000000	1.0000000	1.0000000	0.00000	0.00000	0.00000
1.1	1.1068166	1.1066667	1.1067786	1.1068151	0.14994	0.38019	0.15381
1.2	1.2278796	1.2275679	1.2278006	1.2278764	0.31170	0.79000	0.31947
1.3	1.3641360	1.3636512	1.3640132	1.3641310	0.48472	0.02282	0.49655
1.4	1.5165645	1.5158957	1.5163951	1.5165577	0.66886	0.01694	0.68459
1.5	1.6861706	1.6853072	1.6859520	1.6861618	0.86342	0.02186	0.88320
1.6	1.8739819	1.8729138	1.8737115	1.8739709	0.01068	0.02703	0.01092
1.7	2.0810447	2.0797621	2.0807202	2.0810316	0.01283	0.03245	0.01311
1.8	2.3084212	2.3069148	2.3080401	2.3084058	0.01506	0.03811	0.01539
1.9	2.5571865	2.5554472	2.5567466	2.5571688	0.01739	0.04399	0.1776
2.0	2.8284271	2.8264460	2.8279262	2.8284069	0.01981	0.05009	0.02022

The numerical results become more accurate as the value of h decreases and the error is relatively smaller.

Example 3

Consider the scalar initial value problem

$$y' = -2xy + 4x, \qquad y(0) = 3 \tag{3.3}$$

in the interval $0 \leq x \leq 1$ whose true solution is

$$y(x) = e^{-x^2} + 2.$$

The numerical computation was carried out with three different step sizes $h = 0.1$, $h = 0.025$ and $h = 0.0125$. The numerical results are given in Table 5 below.

Table 5

x_t	Theoretical solution $y(x_t)$	Numerical solution Formula (2.14)			Error		
					$10^{-5} \times E$	$10^{-6} \times E$	$10^{-7} \times E$
		$h = 0.1$	$h = 0.025$	0.0125	$h = 0.1$	$h = 0.025$	$h = 0.0125$
0.0	3.0000000	3.0000000	3.0000000	3.0000000	0.00000	0.00000	0.00000
0.1	2.9900498	2.9900000	2.9924239	2.9912271	0.49834	0.02374	0.01178
0.2	2.9607894	2.9604980	2.9650743	2.9629095	0.29144	0.04285	0.02120
0.3	2.9139312	2.9132415	2.9194869	2.9166738	0.68967	0.05557	0.02743
0.4	2.8521438	2.8509584	2.8582344	2.8551432	0.01185	0.06091	0.0299
0.5	2. 7788008	2.7770952	2.7846859	2.7816914	0.01706	0.05885	0.02891
0.6	2.6976763	2.6955002	2.7026980	2.7001348	0.02176	0.05022	0.02459
0.7	2.6126264	2.6100928	2.6162792	2.6144057	0.02534	0.03653	0.01779
0.8	2.5272924	2.5245578	2.5292668	2.5282426	0.02735	0.01974	0.95020
0.9	2.4448581	2.4420973	2.4450535	2.4449317	0.027608	0.19539	0.73654
1.0	2.3678794	2.3652608	2.3663882	2.3671234	0.02619	0.14913	0.75603

The convergence to the theoretical solution appears to be occurring as the value of h is reduced.

Example 4

Consider the initial value problem

$$y' = e^{10(x-y)}, \quad y(0) = 0.1, \quad h = 0.1 \qquad \text{(Lambert 1973a)} \qquad (3.4)$$

The numerical results are presented in Table 6 below.

Table 6

x_t	Theoretical solution $y(x_t)$	Numerical solution Formula (2.14)	Error $10^{-5} \times E$
0.0	0.1000000	0.10000000	0.000000
0.1	0.1489880	0.14841515	0.057285
0.2	0.2209080	0.22186178	0.095378
0.3	0.3082085	0.31011514	0.001007
0.4	0.4030986	0.40484219	0.001744
0.5	0.5011511	0.50236706	0.001216
0.6	0.6004250	0.60117007	0.074507
0.7	0.7001566	0.70058168	0.042508
0.8	0.8000576	0.80029000	0.023240
0.9	0.9000212	0.90014479	0.012359
1.0	1.0000080	1.0000723	6.430000

These results show a measure of convergence toward the theoretical solution as h tends to zero.

3.1.2 IMPLEMENTATION OF THE PROPOSED METHOD

We now focus our attention on the method that forms the basis of this study. To test the efficiency of the proposed method (2.28), which is the modification of (2.14), we apply the new method to problems described in section (3.1.1) and numerical results are obtained at various step sizes. The numerical results of (2.28) in examples 1, 2, 3 and 4 are presented in tables 7, 8, 9 and 10 respectively. In all the computations we choose $\lambda = 1$.

Table 7

x_t	Theoretical solution $y(x_t)$	Numerical solution Formula (2.28)			Error		
					$10^{-5} \times E$	$10^{-6} \times E$	$10^{-7} \times E$
		$h = 0.2$	$h = 0.1$	$h = 0.05$	$h = 0.2$	$h = 0.1$	$h = 0.05$
0.0	1.0000000	1.0000000	1.0000000	1.0000000	0.000000	0.000000	0.000000
0.1	1.1051709	—	1.1051839	1.10512725	—	0.129849	0.167132
0.2	1.2214028	1.2216354	1.2214315	1.2214063	0.232603	0.287652	0.357826
0.3	1.3498588	—	1.3499066	1.3498647	—	0.477925	0.593827
0.4	1.4918247	1.4923984	1.4918953	1.4918335	0.573750	0.705835	0.875984
0.5	1.64872713	—	1.6488190	1.6487334	—	0.977284	0.012115
0.6	1.8221188	1.8231803	1.8222487	1.8221349	0.01061	0.012990	0.016084
0.7	2.0137527	—	2.0139206	2.0137735	—	0.016787	0.020760
0.8	2.2255409	2.2272869	2.2257534	2.2255672	0.017459	0.021251	0.026249
0.9	2.4596031	—	2.4598679	2.4596358	—	0.026482	0.032672
1.0	2.7182818	2.7209743	2.7186078	2.7183220	0.026925	0.032594	0.040164

As can be seen the proposed method uses much larger stepsizes and still produces better accuracy than the two other methods.

Table 8

x_t	Theoretical solution $y(x_t)$	Numerical solution	Formula (2.28)		Error		
					$10^{-6} \times E$	$10^{-7} \times E$	$10^{-8} \times E$
		$h = 0.1$	$h = 0.05$	$h = 0.01$	$h = 0.1$	$h = 0.05$	$h = 0.01$
1.0	1.0000000	1.0000000	1.0000000	1.0000000	0.00000	0.00000	0.00000
1.1	1.1068166	1.1068540	1.1068252	1.1068169	0.37355	0.85742	0.32559
1.2	1.2278796	1.2279454	1.2278945	1.2278802	0.65811	0.01488	0.55979
1.3	1.3641360	1.3642220	1.3641551	1.3641367	0.86004	0.01908	0.70897
1.4	1.5165645	1.5166630	1.5165858	1.5165653	0.98482	0.02131	0.77867
1.5	1.6861706	1.6862743	1.6861923	1.6861714	0.01037	0.02169	0.77377
1.6	1.8739819	1.8740840	1.8740022	1.8739826	0.01022	0.02032	0.69877
1.7	2.0810447	2.0811389	2.0810620	2.0810452	0.94201	0.01732	0.55733
1.8	2.3084212	2.3085013	2.3084339	2.3084215	0.80165	0.01276	0.35329
1.9	2.5571865	2.5572469	2.5571933	2.5571866	0.60384	0.67339	8.98189
2.0	2.8284271	2.8284623	2.82842640	2.8284269	0.35148	6.95460	0.23013

The errors obtained with this method are smaller than those obtained in Table 4 for the same values of h. This shows that this method is capable of using larger step sizes.

Table 9

x_t	Theoretical solution $y(x_t)$	Numerical solution	Formula (2.28)		Error $10^{-5} \times E$	$10^{-6} \times E$	$10^{-6} \times E$
		$h = 0.1$	$h = 0.025$	0.0125	$h = 0.1$	$h = 0.025$	$h = 0.0125$
0.0	3.0000000	3.0000000	3.0000000	3.0000000	0.00000	0.00000	0.00000
0.1	2.9900498	2.9900295	2.9900459	2.9900487	2.03633	2.32164	1.10095
0.2	2.9607894	2.9605444	2.9607693	2.9607842	0.24502	0.20169	5.23816
0.3	2.9139312	2.9132924	2.9138852	2.9139194	0.63876	0.45975	0.11735
0.4	2.8521438	2.8510027	2.8520664	2.8521242	0.01141	0.77358	0.19579
0.5	2.7788008	2.7771240	2.7786912	2.7787732	0.01677	0.01096	0.27588
0.6	2.6976763	2.6955076	2.6975382	2.6976417	0.02169	0.01381	0.34621
0.7	2.6126264	2.6100764	2.6124673	2.6125866	0.02549	0.01591	0.39745
0.8	2.5272924	2.5245186	2.5271224	2.5272501	0.02774	0.01700	0.42369
0.9	2.4448581	2.4420393	2.4446879	2.4448158	0.028187	0.01701	0.42291
1.0	2.3678794	2.3651902	2.3677194	2.3678397	0.02689	0.01601	0.39693

The numerical results are correctly obtained to four decimal places. These results provide a good approximation as h tends to zero.

Table 10

x_t	Theoretical solution $y(x_t)$	Numerical solution Formula (2.28)	Error
0.0	0.1000000	0.10000000	0.0000000
0.1	0.1489880	0.14954519	0.0055719
0.2	0.2209080	0.22142020	0.0051332
0.3	0.3082085	0.30816102	0.0474800
0.4	0.4030986	0.40281434	0.0028426
0.5	0.5011511	0.50091904	0.0023206
0.6	0.6004250	0.60029353	0.0013147
0.7	0.7001566	0.70009302	0.0635800
0.8	0.8000576	0.80002940	0.0282000
0.9	0.9000212	0.90000928	0.0119200
1.0	1.0000080	1.00000290	0.5100000

The results are correct to five decimal places. This gives an intuitive proof of convergence.

We now consider a different value of λ and see how the solution compare. Numerical results are listed in the tables that follow.

Table 11: Comparison of results for the numerical solution of $y' = y, \quad y(0) = 1$, using formula (2.28) for $\lambda = 0.1$

x_t	Theoretical solution $y(x_t)$	Numerical solution	Formula (2.28)		Error		
					$10^{-6} \times E$	$10^{-7} \times E$	$10^{-8} \times E$
		$h = 0.2$	$h = 0.1$	$h = 0.05$	$h = 0.2$	$h = 0.1$	$h = 0.05$
0.0	1.0000000	1.0000000	1.0000000	1.0000000	0.000000	0.000000	0.000000
0.1	1.1051709	—	1.1051723	1.1051711	—	0.136034	0.177860
0.2	1.2214028	1.2214246	1.2214057	1.2214031	0.218889	0.297558	0.391068
0.3	1.3498588	—	1.3498637	1.3498595	—	0.488196	0.644906
0.4	1.4918247	1.4918761	1.4918318	1.4918256	0.513601	0.712036	0.945361
0.5	1.64872713	—	1.648731	1.6487226	—	0.973685	0.012992
0.6	1.8221188	1.8222093	1.8221316	1.8221205	0.905060	0.012783	0.017141
0.7	2.0137527	—	2.0137690	2.0137549	—	0.016318	0.021988
0.8	2.2255409	2.2256829	2.2255613	2.2255437	0.014196	0.020407	0.027629
0.9	2.4596031	—	2.4596282	2.4596065	—	0.025124	0.034177
1.0	2.7182818	2.7184909	2.7183124	2.7182860	0.029022	0.030552	0.041755

This suggest accuracy to five decimal places for $h = 0.05$. Hence using smaller λ has brought an improvement in the error.

Table 12: Comparison of results for the numerical solution of $y' = xy^{\frac{1}{3}}, \quad y(1) = 1,$

using formula (2.28) for $\lambda = 0.1$

x_t	Theoretical solution $y(x_t)$	Numerical solution Formula (2.28)			Error		
		$h = 0.1$	$h = 0.05$	$h = 0.01$	$10^{-7} \times E$, $h = 0.1$	$10^{-7} \times E$, $h = 0.05$	$10^{-9} \times E$, $h = 0.01$
1.0	1.0000000	1.0000000	1.0000000	1.0000000	0.000000	0.00000	0.00000
1.1	1.1068166	1.1068521	1.1068251	1.1068169	0.035456	0.84535	0.032539
1.2	1.2278796	1.2279414	1.2278942	1.2278802	0.061848	0.01463	0.059387
1.3	1.3641360	1.3642158	1.3641547	1.3641367	0.079812	0.01869	0.070833
1.4	1.5165645	1.5166544	1.5165853	1.5165653	0.089897	0.02076	0.077778
1.5	1.6861706	1.6862632	1.6861916	1.6861714	0.092585	0.02979	0.077262
1.6	1.8739819	1.8740702	1.8740013	1.8739826	0.088308	0.01945	0.069725
1.7	2.0810447	2.0811221	2.0810609	2.0810452	0.077452	0.01259	0.055561
1.8	2.3084212	2.3084815	2.3084327	2.3084215	0.060369	0.01151	0.035126
1.9	2.5571865	2.5572239	2.5571918	2.5571866	0.037378	0.52765	0.874574
2.0	2.8284271	2.8284359	2.8284248	2.8284269	0.877268	0.23659	0.023283

Observe that the accuracy obtained here is higher that achieved in the previous methods.

Table 13: Comparison of results for the numerical solution of

$y' = -2xy + 4x, \quad y(0) = 3$, using formula (2.28) for $\lambda = 0.1$

x_t	Theoretical solution $y(x_t)$	Numerical solution			Error		
			Formula (2.28)		$10^{-6} \times E$	$10^{-8} \times E$	$10^{-9} \times E$
		$h = 0.1$	$h = 0.025$	0.0125	$h = 0.1$	$h = 0.025$	$h = 0.0125$
0.0	3.0000000	3.0000000	3.0000000	3.0000000	0.00000	0.00000	0.00000
0.1	2.9900498	2.9900000	2.9900491	2.9900497	0.49834	0.72842	0.90031
0.2	2.9607894	2.9607013	2.9607882	2.9607893	0.88170	0.01263	0.01556
0.3	2.9139312	2.9138213	2.9139297	2.9139310	0.01099	0.01535	0.01882
0.4	2.8521438	2.8520312	2.8521423	2.8521436	0.01127	0.01521	0.01535
0.5	2.7788008	2.7787039	2.7787995	2.7788006	0.96911	0.01241	0.01497
0.6	2.6976763	2.6976102	2.6976756	2.6976762	0.66111	0.75607	0.89066
0.7	2.6126264	2.6126007	2.6126262	2.6126264	0.25676	0.15271	0.14254
0.8	2.5272924	2.5273104	2.5272929	2.5272925	0.17965	0.47238	0.62729
0.9	2.4448581	2.4449166	2.4448591	2.4448582	0.58541	0.01030	0.01309
1.0	2.3678794	2.3679704	2.3678809	2.3678794	0.90951	0.01453	0.01821

As the h decreases with $\lambda = 0.1$, the proposed method produces results to six-place accuracy. The errors are much smaller than those obtained in Table 9 with $\lambda = 1$.

As indicated in the tables above, the proposed method is capable of using fairly large step sizes and still produces better accuracy than the other methods for differential equations. Although using smaller step sizes, in the above examples, means more computing it has brought an improvement, which seems only fair. It is evident from Tables 7, 8, 9, and 10 that as h tends to zero the computations grow even longer and there is reassuring trend toward the exact values. Tables 11, 12 and 13 give the error for several step sizes and shows that the error decreases when the value of λ is reduced. This emphasizes the dependence of the proposed method on λ. The convergence of the proposed method (2.28) to the exact solution is clearly evident from the above numerical results.

CHAPTER FOUR

4.1 CONCLUDING REMARKS AND DISCUSSION OF THE RESULTS

In chapter three we have developed a method which is stable, consistent and convergent and can favourably handle problems having ~~of~~ exponential solutions. The method compares favourably with existing methods. Computational results in chapter three indicate that the new method (2.28) is accurate and effective for initial value problems with exponential solutions.

The numerical results obtained using the proposed method (2.28) with $h = 0.1$ are better than those obtained with (2.14) for $h = 0.1$, also it is better than those obtained with (2.14) for $h = 0.05$. We also note that this greater accuracy is achieved at the expense of more computation work, since it is now necessary to evaluate $f(x, y)$ three times in order to go from x_t to x_{t+1}. Thus even though more computations are required to use the proposed method (2.28), substantially better results can be obtained with only half as many steps.

The proposed method is of general applicability and it gives a means to compare the accuracy of various numerical methods for solving initial value problems. It can be constructed to have a high degree of accuracy. The quantity λ can be used as a measure of how fast the iteration will converge - the smaller λ is, the faster the convergence. As the estimate tables in section (3.1.2) clearly show, there are two ways to control the error, namely to change h or to change λ. If the error estimate for y_{t+1} is too big, then we should recompute y_{t+1} with smaller value for

h or with smaller value for λ. On the other hand if our error estimate is extremely small, then efficiency considerations suggest that λ should be increased.

This accuracy is clearly indicated by the results tabulated in section 3.1.2 which are in fairly close agreement with the values of the theoretical solution.

Finally, our experiments with this method do not lead us to believe that the last word on the subject has been said; however, we have encountered enough complexities to convince us that the stage we have reached represents a good compromise between economy and accuracy, which is likely to be acceptable in many applications.

4.2 IMPLEMENTATION PROBLEMS

The mathematical formulation of physical phenomena in simulation, electrical engineering, control theory, and economics often leads to an initial value problem of the form (1.11) in which there is a pole in the solution. The following are examples of initial value problems in which there are poles in the solution

As an example consider the initial value problem

$$y' = y^2, \qquad y(0) = 1, \quad 0 < x \leq 1 \tag{4.1}$$

whose theoretical solution is $y_t(x) = \frac{1}{1-x}$.

Application of our method (2.28) to the initial value problem (4.1) leads to an instability near the pole. Equally, some of the difficulties that were encountered during implementation is the application of (2.28) to problems of the form

$$y' = 1 + y^2, \qquad y(0) = 1, \qquad h = 0.25, \quad 0 \leq x \leq 1 \tag{4.2}$$

whose exact solution curve is $y_2(x) = \tan(x + \frac{\pi}{4})$ and pole at $x = \frac{\pi}{4}$.

Thus $y_1(x) \to \infty$ as $x \to 1$ and $y_2(x) \to \infty$ as $x \to \pi/4 \approx 0.785$. These calculations show that the solutions of original initial value problems must be unbounded somewhere at $x = 1$ and $x \approx 0.785$. However, our numerical calculations suggest that we can go very near $x = 0.785$ and probably $x = 0.9$.

4.3 POSSIBLE EXTENSION

The purpose of this section is to provide an overview of some of the areas of possible extension of this work to incorporate solutions of problems with singularity or poles. Of particular interest is the performance of the proposed method on initial value problems with points of singularity. Such problems generally require a combination of analytical and numerical work. Since the present method cannot cope with the above problems, this difficulty forms the area of possible extension of the present work. Other areas that can benefit from further study include stability and convergence rate and also generalisation of the proposed method to give a class of one-step methods.

4.4 AREAS OF APPLICATIONS

Ordinary differential equations have been studied by mathematicians and Physicists over the years. They arise most commonly in the study of physical phenomena. Numerical techniques such as the one developed here, can be exploited in different application areas. For instance, the one-step method we proposed can play a significant role in the solution of non-stiff ordinary differential equations arising in electrical and electronic networks, economic system models, radioactive processes, population and ecological models and other dynamic processes such as diffusion, heat transfer and pharmaco-kinetic theories.

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