UNIVERSITY OF FORT HARE

STM 312

DEGREE EXAMINATIONS
JUNE 2017

Time : 3 hours

Subject : Advanced Mathematical Statistics

Marks : 100

This paper consists of 9 printed pages including the cover page

Internal Examiners

Mr R.T Chiruka

External Examiners

Prof DJ. De Waal

Instructions

Answer all questions

Start Each Question on a Fresh Page

Statistical Tables will be provided

QUESTION ONE [32]

1.1. If the joint probability distribution of X_1 and X_2 is given by

$$f(x_1, x_2) = \begin{cases} \dfrac{x_1 x_2}{36} & \text{for } x_1 = 1,2,3; x_2 = 1,2,3 \\ 0 & \text{elsewhere} \end{cases}$$

Find

a. The probability distribution of $X_1 X_2$. [2]

b. The probability distribution of $\dfrac{X_1}{X_2}$. [2]

c. The joint distribution of $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$. [5]

d. The marginal distribution of Y_1. [2]

e. Find $\text{cov}(Y_1, Y_2)$. [5]

1.2. Suppose that $X_1, X_2, \ldots, X_n$ are independent random variables each having a Poisson distribution with mean λ.

a. Find the distribution of $\sum_{i=1}^{k} X_i$. [3]

b. Find the distribution of $\sum_{i=1}^{k} X_i$. When the λ_i changes from X_i to X_j. (that is the X_i have different λ.) [3]

1.3. Let Y_1 and Y_2 be independent random variable each with an exponential distribution with mean 1 (one). Let $Z_1 = Y_1 + Y_2$ and $Z_2 = Y_1 - Y_2$.

a. Find the joint probability density function of Z_1 and Z_2. [6]

b. Find the marginal density function of Z_2. [3]

c. Are Z_1 and Z_2 independent? [1]

QUESTION TWO [24]

The management of fast food outlet is interested in the joint behaviour of the random variables Y_1, defined as the total time between a customer's arrival at the store and the departure from the service window, and Y_2, the time a customer waits in line before reaching the service window. Because Y_1 includes the time a customer waits in line we must have $Y_1 \geq Y_2$. The relative frequency of observed values of Y_1 and Y_2 can be modelled by the probability density function

$$f(y_1, y_2) = \begin{cases} e^{-y_1} & \text{for } 0 \leq y_2 \leq y_1 \leq \infty \\ 0 & \text{elsewhere} \end{cases}$$

With time measured in minutes.

a. Find $P(Y_1 < 2, Y_2 > 1)$. [3]

b. Find the probability that the time spent at the service window is more than one minute. [4]

c. Find the conditional density function of Y_1 given that $Y_2 = y_2$. Specify the values of y_2 for which this conditional density is defined. [4]

d. Are Y_1 and Y_2 independent? [2]

e. The random variable $Y_1 - Y_2$ represents the time spent at the service window. find

   i. $E(Y_1 - Y_2)$. [4]

   ii. $\text{var}(Y_1 - Y_2)$. [7]

QUESTION THREE [20]

Let $X_1, X_2, \ldots\ldots X_n$ be independent uniformly distributed random variables on the interval $[0,\theta]$. Let $Y_1 < Y_2 < \ldots\ldots. < Y_n$ denote the order statistic for this random sample . Find

a. The density function of $Y_{(k)}$, the k^{th} order statistic. [5]

b. If $\theta = 1$ show that $Y_{(k)}$, has a Beta density function with $\alpha = k$ and $\beta = n - k + 1$. [4]

c. If $\theta = 1$ find $\operatorname{var}\left(Y_{(k)}\right)$. [3]

d. If $j = 1$ and $k = n$ show that $U_1 = \dfrac{Y_{(1)}}{Y_{(n)}}$ and $U_2 = Y_{(n)}$ are independent [8]

QUESTION FOUR[11]

4.1 prove the following theorem:

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a normal distribution with mean μ and variance σ^2 . then

$$\frac{(n-1)S^2}{\sigma^2}$$

Has a $\chi^2_{(n-1)}$ distribution. [5]

4.2 random variables $X_1, \ldots\ldots, X_{100}$ are all independent and all have the normal distribution with mean 0 and variance 1. Identify the distribution of Y in each of the cases below. Give reasons for your choice of distribution.

a. $Y = \sum_{i=1}^{10} X_i^2$ [2]

b. $Y = \dfrac{X_{51}}{\sqrt{\sum_{i=1}^{50} X_i^2 / 50}}$ [2]

c. $Y = \dfrac{\sum_{i=51}^{98} X_i^2 / 48}{\sum_{i=1}^{50} X_i^2 / 50}$ [2]

QUESTION FIVE [13]

5.1 State central limit theorem. [3]

5.2 State and prove the weak law of large numbers. [5]

5.3 Let Y_n have a distribution that is $b(n, p)$. Suppose that the mean $\mu = np$ is the same for every n, that is $p = \frac{\mu}{n}$ where μ is a constant. Find the limiting distribution when $p = \frac{\mu}{n}$. [5]

APPENDIX A

$$g_k\left(y_{(k)}\right)=\frac{n!}{(k-1)!(n-k)!}\left[F\left(y_{(k)}\right)\right]^{k-1}\left[1-F\left(y_{(k)}\right)\right]^{n-k} f\left(y_{(k)}\right) \quad \text{for } -\infty < y_{(k)} < \infty$$

If j and k are 2 integers such that $1 \leq j < k < n$ then

$$g_{j,k}(y_j, y_k)=\frac{n!}{(j-1)!(k-1-j)!(n-k)!}\left[F(y_j)\right]^{j-1} * \left[F(y_k)-F(y_j)\right]^{k-1-j} * \left[1-F(y_k)\right]^{n-k} f(y_j)f(y_k) \quad -\infty < y_j < y_k < \infty$$

Continuous Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Uniform	$f(y) = \frac{1}{\theta_2 - \theta_1}; \theta_1 \le y \le \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$
Normal	$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\left(\frac{1}{2\sigma^2}\right)(y-\mu)^2\right]$ $-\infty < y < +\infty$	μ	σ^2	$\exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$
Exponential	$f(y) = \frac{1}{\beta}e^{-y/\beta}; \quad \beta > 0$ $0 < y < \infty$	β	β^2	$(1 - \beta t)^{-1}$
Gamma	$f(y) = \left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\right] y^{\alpha-1}e^{-y/\beta};$ $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$	$(1 - \beta t)^{-\alpha}$
Chi-square	$f(y) = \frac{(y)^{(v/2)-1}e^{-y/2}}{2^{v/2}\Gamma(v/2)};$ $y^2 > 0$	v	$2v$	$(1-2t)^{-v/2}$
Beta	$f(y) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] y^{\alpha-1}(1-y)^{\beta-1};$ $0 < y < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	does not exist in closed form