

UNIVERSITY OF FORT HARE

Complex Analysis
MAT 322

Examination

November

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Time: 3 hours

Subject: Complex Analysis

Maximum Marks: 100



This question paper consists of 4 pages

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Internal examiner

Prof A.L. Prins

External Examiner

Prof T.T. Seretlo

Instructions

Answer Five Questions

Symbols have the usual meanings.

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QUESTION 1

Choose at least **one correct answer** and **write your answer on the question paper and hand it in with your examination book.**

1.1) The **image** of the disk $|z - 1| < 1$ under the transformation $w = (3 + 4i)z - 2 + i$ is given by

- (A) $|w - 1 + 5i| > 5$ (B) $|w + 1 + 5i| < 5$ (C) $|w - (1 + 5i)| < 5$ (D) $|w - 1 + 5i| < 5$
 (E) None of the above.

1.2) $\lim_{z \rightarrow i} \frac{z^4 - 1}{z - i}$

- (A) 4 (B) $1 - 4i$ (C) -4 (D) $-4i$ (E) None of the above.

1.3) The function $f(z) = |z|^2$ is differentiable on ...

- (A) $\{x + iy | x \in \mathbb{R}, y = 0\}$ (B) $\{x + iy | x = 0, y = 0\}$ (C) complex plane (D) nowhere
 (E) None of the above.

1.4) $\lim_{n \rightarrow \infty} \left(\frac{1}{2} + \frac{i}{4}\right)^n$

- (A) Is divergent. (B) converges to $\frac{1}{2} + \frac{i}{4}$. (C) converges to 1. (D) converges to 0.
 (E) None of the above.



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1.5) $\sum_{n=0}^{\infty} \left(\frac{1+i}{3}\right)^n =$

- (A) $\frac{6+3i}{5}$ (B) $\frac{6+3i}{\sqrt{5}}$ (C) $\frac{6-3i}{5}i$ (D) $\frac{-6-3i}{5}i$ (E) None of the above.

1.6) The disk of convergence for the series $\sum_{n=0}^{\infty} (z + i)^n (6 + 8i)^{-n}$ is

- (A) $|z - i| < 10$ (B) $|z + i| < 100$ (C) $|6 + 8i| < 5$ (D) $|z + i| < 1$ (E) None of the above.

1.7) Find all values of $z \in \mathbb{C}$ for which $e^z = -1 + i$ hold.

- (A) $\ln(\sqrt{2}) + i\left(\frac{\pi}{4}\right)$ (B) $\ln(\sqrt{2}) + i\left(\frac{\pi}{4} + 2\pi n\right), n \in \mathbb{Z}$ (C) $\ln(\sqrt{2}) + i\left(\frac{3\pi}{4} + 2\pi n\right), n \in \mathbb{Z}$ (D) none
 (E) None of the above.

1.8) Find the principal value of $i^{0.5i}$.

- (A) $e^{-\left(\frac{\pi}{4} + 2\pi n\right)}, n \in \mathbb{Z}$ (B) $e^{-i\left(\frac{\pi}{2} + 2\pi n\right)}, n \in \mathbb{Z}$ (C) $e^{-i\frac{\pi}{2}}$ (D) $e^{-\frac{\pi}{2}}$ (E) None of the above.

1.9) $\int_0^1 \frac{t}{t+i} dt =$

- (A) $1 + \arctan(1) - i\ln(\sqrt{2})$ (B) $\frac{\pi}{4} - i\ln\sqrt{2}$ (C) $2 + i\ln\sqrt{5}$ (D) $1 - \frac{\pi}{4} - i\ln\sqrt{2}$
(E) None of the above.

1.10) Evaluate $\int_C z dz$ where C is the line segment from i to 1 given by the parametrization $z(t) = 1 - t + it, 0 \leq t \leq 1$.

- (A) -1 (B) $-i$ (C) i (D) 1 (E) None of the above.

[10x3=30]

QUESTION 2

2.1) Explain whether $\lim_{z \rightarrow 0} \frac{\bar{z}}{z}$ exists.

[4]

2.2) Let $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ be differentiable at the point $z_0 = x_0 + iy_0$. Prove that the partial derivatives of u and v exist at the point (x_0, y_0) and satisfy the equations $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$.

[7]

2.3) Consider the harmonic function $v(x, y) = x^3 - 3xy^2$. Construct a harmonic conjugate $u(x, y)$ of $v(x, y)$ such that $f(z) = u(x, y) + iv(x, y)$ is analytic.

[6]

2.4) Can you find an analytic function $f(z) = u(x, y) + iv(x, y)$ such that $v(x, y) = x^3 + y^3$? Explain.

[3]

[20]

QUESTION 3

3.1) Use $\int_C \frac{1}{z - z_0} dz = 2\pi i$, where C is a simple closed contour with positive orientation

such that z_0 lies interior to C together with the Cauchy-Goursat theorem to

compute $\int_C \frac{5z-1}{2z^2+z-1} dz$ where $C: |z+1| = 1$.

[7]

3.2) Use the **Cauchy Integral Formula** to compute $\int_C \frac{e^z}{z^2+4} dz$ along the positive oriented contour $C: |z+2i|=3$. [6]

3.3) Use the **Cauchy Integral Formulae for derivatives** to compute $\int_C \frac{1}{(z^2-zi)^2} dz$ along the positive oriented contour $C: |z-i|=\frac{1}{2}$. [6]

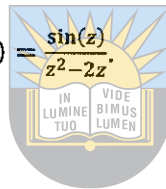
[19]

QUESTION 4

4.1) Use a **geometric series** to compute the **Taylor series** for $f(z) = \frac{1-z}{z-2}$ and its **region of convergence**. [6]

4.2) Compute the **Laurent series** representation for $f(z) = \frac{1}{2+z-z^2}$ involving **powers of z** in the domain $1 < |z| < 2$. [6]

4.3) **Classify the singularities** (if any) of $f(z) = \frac{\sin(z)}{z^2-2z^*}$. [3]



[15]

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QUESTION 5

5.1) Let D be a simply connected domain and let C be a simple closed positively oriented contour that lies in D . If f is analytic inside C and on C , except at the points $z_1, z_2, \dots, z_n$ that lie inside C , then prove $\int_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f, z_k)$. [5]

5.2) Use Cauchy's Residue Theorem to compute $\int_C \frac{1}{z(z^2-2z+2)} dz$, where $C: |z-i|=2$ [7]

5.3) Use **residues** to compute $\int_{-\infty}^{\infty} \frac{1}{x^2+4} dx$. [5]

[17]

END
