UNIVERSITY OF FORT HARE

Integral Calculus: A Theoretical Approach
Mat 121

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This question paper consists 4 pages

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Instructions
Answer all questions.
Symbols used have the usual meaning
**Question 1**

1.1 For the complex number \( z = 2 - i\sqrt{7} \). Find:
- (a) the conjugate (1)
- (b) the modulus (1)
- (c) the argument (1)

1.2 Prove that \( \frac{z_1 z_2}{\bar{z}_1 \bar{z}_2} \) for complex numbers \( z_1 \) and \( z_2 \). (3)

1.3 State without proof De Moivre’s Theorem.
Hence use this theorem to find \( (2\sqrt{3} + 2i)^5 \). (5)

1.4 Find the cube root of \( 8i \) and sketch the roots on the \( z \)-plane. (4)

1.5 For \( u = x^6 yz \) find \( \frac{\partial^2 u}{\partial x \partial y} \). (2)

**Question 2**

2.1 First use substitution and then integration by parts to evaluate the integral \( \int \sin \sqrt{x} dx \). (4)

2.2 Evaluate the integrals
   - (a) \( \int \tan^2 x \sec^4 x dx \)
   - (b) \( \int \cos 3x \sin 6x dx \) (6)

2.3 Chose a suitable trigonometric substitution to evaluate the following integrals:
   \( \int \frac{\sqrt{9x^2 - 4}}{x} dx \) (5)

2.4 Write down the partial decomposition of \( \frac{1}{x(x+1)(2x+3)} \). After evaluating for the constants use this partial decomposition to evaluate \( \int \frac{1}{x(x+1)(2x+3)} dx \). (4)

2.5 Use an appropriate rationalizing substitution to evaluate the integral
   \( \int \frac{1}{\sqrt{x} + \sqrt[4]{x}} dx \) (4)

2.6 An integrating factor \( I(x) \) is used to solve the linear differential equation
   \( \frac{dy}{dx} + P(x)y = Q(x) \). Use this procedure to find the integrating factor for the initial value problem \( x^4 \frac{dy}{dx} + 2xy = \cos x \), \( y(\pi) = 0 \). (3)
2.7 Determine whether the equation \( y' = \frac{xy + y^2}{x^2} \), is homogeneous, and solve it. (4)

2.8 Show that the equation \( \sin y + (1 + x \cos y) \frac{dy}{dx} = 0 \) is exact, and hence solve it. (4)

**Question 3**

3.1 If \( A \) is a \( m \times n \) matrix and \( B = A^T \), find the size of

(a) \( B \)
(b) \( BB^T \)
(c) \( B^T B \) (3)

3.2 Given the matrix \( A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \\ 2 & 0 & 1 \end{bmatrix} \)

(a) Find \( \det(A) \) by

(i) Cofactor expansion method.
(ii) “Basket-Weave” method. (4)

(b) Find \( A^{-1} \) using the Adjoint method. (5)

\[ x - z = 3 \] (1)

3.3 (a) Write the system of linear equations \( 2y - 2z = 2 \) in matrix form.
\[ 2x + z = 3 \]

(b) Solve the system using **Cramer’s Rule**. (4)

3.4 Find all the solutions of the given linear system using the Gauss method with back substitution, \( x_1 - x_2 + x_3 = 7 \).
\[ 2x_1 + 2x_2 + x_3 = 4 \] (4)

**Question 4**

4.1 The ellipse \( \left( \frac{x^2}{16} \right) + \left( \frac{y^2}{9} \right) = 1 \) is shifted 4 units to the right and 3 units up to generate the ellipse \( \left( \frac{(x-4)^2}{16} \right) + \left( \frac{(y-3)^2}{9} \right) = 1 \).

(a) Find the foci, vertices, and center of the new ellipse. (3)

(b) Plot the new foci, vertices, and center, and sketch in the new ellipse. (4)
4.2 Given that $P_1(1,-1,3)$ and $P_2(-1,4,5)$ find
(i) The coordinates of the midpoint of the line segment joining $P_1$ to $P_2$.
(ii) The unit vector in the direction of $\overrightarrow{P_1P_2}$.
(iii) Express $\overrightarrow{P_1P_2}$ as a product of its length and direction.
(iv) Find the angle between the vectors $\mathbf{u} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{k}$.
Are they orthogonal? (6)

**Question 5**

5.1 Show that $\tanh(\ln x) = \frac{x^2 - 1}{x^2 + 1}$. (3)

5.2 Find $\frac{dy}{dx}$ for the following $y = x^{\sin x}$. (3)

5.3 Let $a_n = \frac{2n}{3n + 1}$
(a) Determine whether $\{a_n\}$ is convergent. (3)
(b) Determine whether $\{a_n\}$ is convergent. (3)

5.4 Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence. (3)