University of Fort Hare

MAT 303

Supplementary Examinations: January 2019
Subject: Mathematics 3
Paper: Real Analysis

Time: 3 Hours  Marks: 100  Subminimum: 40

This question paper consists of 5 pages

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Instructions

Attempt NO more than FIVE(5) questions. Symbols used have the usual meanings.

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Mathematics Supplementary Exams November 2018

Course Code: MAT 303
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**Question One**

1.1 (a) Define a metric on a non-empty set $X$. (2)
(b) Define the diameter of a set $A$ in $\mathbb{R}^n$. Hence find the diameter of the set $\{ (x, y) : \frac{x^2}{25} + \frac{y^2}{25} \leq 1 \}$ in $\mathbb{R}^2$. (3)
(c) Define the trivial metric on $X$. (1)
(d) Define the usual metric on $\mathbb{R}^n$. (1)

1.2 Let $d$ be a metric on $X$, $A \subset X$ and $p \in X$. The distance between $p$ and $A$ is $d(p, A) = \inf\{d(p, a) : a \in A\}$. Let $d$ be the trivial metric on $\mathbb{R}$ and $A = (-2,5) \subseteq \mathbb{R}$. (a) Find $d(5,A)$. If $\rho$ is the usual metric on $\mathbb{R}$, (b) find (with motivation) (i) $\rho(5, A)$ and (ii) $\rho(-3,A)$. (3)

1.3 Show that, in $\mathbb{R}^2$, the set $A = \{ (x,y) : 0 < y < 3 \}$ is open. Draw also a rough sketch of set $A$. (4)

1.4 (a) Define the boundary of a subset $A$ of $\mathbb{R}^n$. Then give another characterization of a boundary. (2)
(b) Let $A = \{ (x,y) \in \mathbb{R}^2 : -1 < x \leq 1 \}$. Find the boundary of $A$. Draw set $A$. (3)
(c) Show that a boundary point need not be an accumulation point. (2)

1.5 (a) Show that the set $A = \{ 1 + \frac{1}{n} : n \in \mathbb{N} \}$ is not closed in $\mathbb{R}$. (2)

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Question Two

2.1 If \( x = \sup S \), for \( S \subseteq \mathbb{R} \), show that \( x \in \overline{S} \). \( (3) \)

2.2 Prove that a set \( A \subseteq \mathbb{R}^n \) is closed if and only if for every sequence \( \{x_k\} \) in \( A \) which converges, the limit lies in \( A \). \( (4) \)

2.3 Let \( B \subseteq \mathbb{R}^n \). Prove that \( x \in \overline{B} \) if and only if there is a sequence \( \{x_k\} \) in \( B \) converging to \( x \), where \( \overline{B} \) is the closure of \( B \) in \( \mathbb{R}^n \). \( (5) \)

2.4 A metric space \((M, d)\) is complete if every Cauchy sequence in \( M \) converges to a point in \( M \). Show that the set \( Q^c \) of all irrational numbers (with the usual metric) is not a complete metric space. \( (2) \)

2.5 (a) Let \( A \subseteq \mathbb{R} \), \( x, y \in A \). Define what is meant by a path joining \( x \) to \( y \). \( (1) \)

(b) Let \( A \subseteq \mathbb{R}^n \). Define what is meant by (i) \( A \) is compact, (give three equivalent statements), (ii) \( A \) is path-connected. \( (4) \)

2.6 Show that \( A = \{x \in \mathbb{R}^n : ||x|| \leq 4\} \) is path-connected. \( (3) \)

Question Three

3.1 (a) The set \( S = \{(x, y) \in \mathbb{R}^2 : -1 < x < 2\} \) is not compact. Why? \( (1) \)

(b) Let \( A = \{1 + \frac{1}{n} : n = 1, 2, \ldots \} \). Is \( A \) compact? Explain. If \( A \) is not compact, how can we “compactify” it? \( (2) \)

3.2 The set \( \{(1 + \frac{1}{n}, 3) : n = 1, 2, \ldots \} \) is an open cover of \((1, 2)\). Show that this cover has no finite subcover of \( A \). \( (4) \)

3.3 Show that \([2, 3] \cap Q\) is not path-connected, where \( Q \) is the set of all rational numbers. \( (3) \)

3.4 Prove that a set \( A \subseteq \mathbb{R} \) is connected if and only if \( A \) is an interval. \( (4) \)
3.5 Let \( f : A \to \mathbb{R}^n \). Prove that the following statements are equivalent:
(i) \( f \) is continuous on \( A \);
(ii) For each convergent sequence \( \{x_n\} \), converging to \( x_0 \in A \), the sequence \( \{f(x_n)\} \) converges to \( f(x_0) \);
(iii) For each open set \( U \subset \mathbb{R}^n \), \( f^{-1}(U) \subset A \) is open relative to \( A \), i.e. \( f^{-1}(U) = V \cap A \) for some open set \( V \) in \( A \).

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**Question Four**

4.1 Let \( f : A \to \mathbb{R}^n \) be a continuous function. Prove that
(a) If \( K \subset A \) and \( K \) is connected, then \( f(K) \) is connected.
(b) If \( B \subset A \) and \( B \) is compact, then \( f(B) \) is compact.

4.2 Let \( f(x) = \frac{1}{x} \) where \( f : (0, \infty) \to \mathbb{R} \). Show, from definition, that \( f \) is continuous at \( x_0 \in (0, \infty) \).

4.3 Let \( f : [0, 1] \to [0, 1] \) be continuous. Show that \( f \) has a fixed point.

4.4 (a) Find a continuous map \( f : \mathbb{R} \to \mathbb{R} \) and a compact set \( K \subset \mathbb{R} \) such that \( f^{-1}(K) \) is not compact
(b) Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a continuous function and let \( A = \{f(x) : ||x|| = 3\} \). Show that \( A \) is a closed interval.

4.5 Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be continuous. Show that \( \{x \in \mathbb{R}^n : ||f(x)|| < 2\} \) is open in \( \mathbb{R}^n \).

4.6 Let \( \{f_k : A \to \mathbb{R}^m\} \) be a sequence of functions.
(a) Define what is meant by the sequence of functions converges uniformly to \( f \).
(b) Let \( \{f_k : \mathbb{R} \to \mathbb{R}\} \) be given by \( f_k(x) = \begin{cases} 0 & \text{if } x < k \\ 1 & \text{if } x \geq k \end{cases} \). Show that the sequence \( \{f_k\} \) converges to 0 but the convergence is not uniform.
Question Five

5.1 Let \( \{ f_k : A \to \mathbb{R}^m \} \) be a sequence of continuous functions, and suppose \( f_k \to f \) uniformly. Prove that \( f \) is continuous. \( \quad (4) \)

5.2 (a) State, without proof, the Weierstrass M-test for uniform convergence. \( \quad (2) \)

(b) Show that \( \sum_{n=1}^{\infty} \frac{(\sin(nz))^2}{n^3} \) converges uniformly. \( \quad (2) \)

5.3 (a) State, without proof, the Contraction Mapping Principle. \( \quad (2) \)

(b) Give an example of a complete metric space \( X \) and a map \( T : X \to X \) with \( d(T(x), T(y)) \leq d(x, y) \) but having no fixed point. \( \quad (3) \)

5.4 (a) Let \( f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m \). Define what is meant by \( f \) is differentiable at \( x_0 \in A \). \( \quad (2) \)

(b) Define the Jacobian matrix of the function \( f \) in (a) above. \( \quad (3) \)

(c) Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be given by \( f(x, y) = (x^3, x^3y, x^3y^2) \). Compute the Jacobian matrix of \( f \). \( \quad (2) \)

5.5 Let \( x, y \in R \). Let \( U(x, y) = \frac{x^4 + y^4}{4} \) and \( V(x, y) = \cos x + \sin y \). Find at least one point near which we can solve for \( x, y \) in term of \( U, V \). \( \quad (4) \)

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Question Six

6.1 Let \( x, y \in R \). Let \( U(x, y) = e^x \sin y \) and \( V(x, y) = e^x \cos y \). Show that \( f(x, y) = (U(x, y), V(x, y)) \) is locally invertible, but NOT invertible. \( \quad (4) \)

6.2 (a) Define a partition of an interval \([a, b]\). (b) Define the upper and the lower sums of a function \( f : [a, b] \to [0, \infty] \). (c) Define what is meant by \( f \) is Riemann integrable. \( \quad (5) \)

6.3 Let \( Q^c \) and \( Q \) denote the sets of irrational and rational numbers respectively. Let \( f : [0, 1] \to R \) be given by \( f(x) = \begin{cases} 
1 & \text{if } x \in Q^c \\
0 & \text{if } x \in Q
\end{cases} \) Show that \( f \) is NOT Riemann integrable. \( \quad (4) \)
6.4 (a) Define the volume of a subset $A$ of $\mathbb{R}^n$. (b) If $A = [3, 5] \times [-1, 2]$ and $n = 2$, find the volume of $A$. (c) Define what is meant by a subset $A$ of $\mathbb{R}^n$ has measure zero. (4)

6.5 (a) State, without proof, Lebesgue's Theorem. (b) State a corollary of the Lebesgue Theorem that characterises a set of measure zero in terms of volume. (4)

6.6 Let $A$ be a set and $\mathcal{P}(A)$ the power set of $A$. Prove that $|A| < |\mathcal{P}(A)|$. (4)

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