UNIVERSITY OF FORT HARE

Integral Calculus: A Theoretical Approach
Mat 121

Degree Examinations

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Time: 3 Hours
Subject: MAT 121
Marks: 100

This question paper consists of 3

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Instructions
Answer all questions.
Symbols used have the usual meaning
**Question 1**

1.1 Express the complex number \( z = \frac{3}{4 - 3i} \), in the form \( a + ib \).

1.2 Prove that \( \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \) for complex numbers \( z_1 \) and \( z_2 \).

1.3 State without proof De Moivre’s Theorem. Hence use this theorem to find \((1 - \sqrt{3}i)^5\).

1.4 Find the cube roots of \( i \) and sketch the roots on the \( z \)-plane.

1.5 For \( z = x \sin y \) find \( \frac{\partial^3 z}{\partial y^2 \partial x} \).

**Question 2**

2.1 First use substitution and then integration by parts to evaluate the integral \( \int x^3 e^{x^2} \, dx \).

2.2 Evaluate the integrals (a) \( \int \cot^2 x \, dx \) (b) \( \int \cos 3x \sin 6x \, dx \).

2.3 Chose a suitable trigonometric substitution to evaluate the following integrals:

\[ \int_0^1 x^2 \sqrt{9 - x^2} \, dx \]

2.4 Write down the partial decomposition of \( \frac{1}{x(x+1)(2x+3)} \). After evaluating for the constants use this partial decomposition to evaluate \( \int \frac{1}{x(x+1)(2x+3)} \, dx \).

2.5 Use an appropriate rationalizing substitution to evaluate the integral \( \int \frac{1}{3 \sin x + 4 \cos x} \, dx \).

2.6 An integrating factor \( I(x) \) is used to solve the linear differential equation \( \frac{dy}{dx} + P(x)y = Q(x) \). Use this procedure to find the integrating factor for the initial value problem \( x^2 \frac{dy}{dx} + 2xy = \cos x, y(\pi) = 0 \).

2.7 Determine whether the equation \( y' = \frac{xy + y^2}{x^2} \), is homogeneous, and solve it.
2.8 Show that the equation $\sin y + (1 + x \cos y) \frac{dy}{dx} = 0$ is exact, and hence solve it. (4)

**Question 3**

3.1 If $A$ is a $m \times n$ matrix and $B = A^T$, find the size of
   (a) $B$
   (b) $BB^T$
   (c) $B^TB$ (3)

3.2 Given the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \\ 2 & 0 & 1 \end{bmatrix}$
   (a) Find $\det(A)$ by
      (i) Cofactor expansion method.
      (ii) “Basket-Weave” method. (4)
   (b) Find $A^{-1}$ using the Adjoint method. (5)

   $$x - z = 3$$

3.3 (a) Write the system of linear equations $2y - 2z = 2$ in matrix form. (1)
   $$2x + z = 3$$
   (b) Solve the system using Cramer’s Rule. (4)

3.4 Find all the solutions of the given linear system using the Gauss method with back
   substitution, $x_1 - x_2 + x_3 = 7$. (4)
   $$2x_1 + 2x_2 + x_3 = 4$$

**Question 4**

4.1 Find the equation of the conic, hyperbola that satisfies the conditions;
   the foci $(\pm6,0)$ vertices $(\pm4,0)$ and sketch its graph. (7)

4.2 Given that $P_1(1,-1,3)$ and $P_2(-1,4,5)$ find
   (i) The coordinates of the midpoint of the line segment joining $P_1$ to $P_2$.
   (ii) The unit vector in the direction of $P_1P_2$.
   (iii) Find the angle between the vectors $u = 2i + 6j - 4k$
       and $v = -3i - 9j + 6k$. Are they orthogonal, parallel or neither? (6)
Question 5

5.1 Show that $\sinh^{-1} x = \frac{1}{\sqrt{1 + x^2}}$.

5.2 Find $\frac{dy}{dx}$ for the following $y = e^{\tanh x} \cosh(\cosh x)$.

5.3 Let $a_n = \frac{4n - 3}{3n + 4}$, determine whether $\{a_n\}$ is convergent or diverges.