University of Fort Hare

MAT 224

Degree Examinations: November 2018
Subject: Mathematics 2
Paper: Real Analysis

Time: 3 Hours  Marks: 100  Subminimum: 40

This question paper consists of 4 pages

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Instructions
Answer NO more than FIVE (5) questions. Symbols used have the usual meanings.

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Mathematics Exams November 2018

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Question One

1.1 Let $A$, $B$ and $C$ be sets. Prove that (a) $A - (B \cup C) = (A - B) \cap (A - C)$;
   (b) $A \subseteq B$ if and only if $A \cap B^c = \emptyset$, where $A^c = X - A$.  (6)

1.2 If $A, B \subseteq X$, we write $B^c$ for $X - B$ and $A^c$ for $X - A$. Prove that $A \subseteq B$ iff $B \cup A^c = X$.  (3)

1.3 (a) Define what is meant by “set $A$ is equivalent (equipotent) to set $B$”.
   (1)

   (b) Show (from definition) that the interval $(0,1)$ is equivalent to the set $R$ of all real numbers.  (4)

1.4 (a) Define what is meant by a “countably infinite set” and hence define a countable set.
   (2)

   (b) Prove that the set $Q$ of all rational numbers is countable.  (4)

1.5 (a) Let $S$ be a bounded set. Define what is meant by (i) the supremum of $S$ and (ii) the infimum of $S$.
   (3)

   (b) Let $S = \{x : 3x^2 - 5x \leq 2\}$. Find (i) sup $S$ and inf $S$, if they exist.  (3)

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Question Two

2.1 Prove the following: Let $S \subseteq \mathbb{R}$. A real number $a$ is the least upper bound of $S$ (i.e. $a = \sup S$) if and only if $a$ is an upper bound of $S$ and for every $\epsilon > 0$ there exists $x \in S$ such that $a - \epsilon < x$.  (6)
2.2 Let $x$ be a real number. Prove that there exists an integer $n$ such that $n \leq x < n + 1$. (4)

2.3 Prove the theorem of Eudoxus: Between any two real numbers there is a rational number. (5)

2.4 (a) Let $\varepsilon > 0$. Prove that $|a| < \varepsilon$ if and only if $-\varepsilon < a < \varepsilon$. (3)
(b) Let $x, y, a, b \in \mathbb{R}^+$ and suppose $\frac{x}{y} < \frac{a}{b}$. Prove that $\frac{x}{y} < \frac{x+a}{y+b} < \frac{a}{b}$. (3)

2.5 Prove that $||a| - |b|| \leq |a - b|$ for any real numbers $a$ and $b$. (3)

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**Question Three**

3.1 (a) Define what is meant by “a subset $A$ of $\mathbb{R}$ is open in $\mathbb{R}$”. (1)
(b) Let $a \in \mathbb{R}$. Prove that the interval $(-\infty, a)$ is an open subset of $\mathbb{R}$. (3)
(c) Let $a, b \in \mathbb{R}$ with $a < b$. Show that the interval $(a, b]$ is NOT an open subset of $\mathbb{R}$. (2)

3.2 (a) Prove that the union of any number of open sets in $\mathbb{R}$ is open. (3)
(b) Prove that the intersection of a finite number of open sets in $\mathbb{R}$ is open. (3)

3.3 Show from definition that the interval $[a, \infty)$ is closed in $\mathbb{R}$. (2)

3.4 (a) Define what is meant by “$x$ is an interior point of $A$”. (1)
(b) Let $A \subseteq \mathbb{R}$. Prove that $A^o$ is an open subset of $\mathbb{R}$. (4)

3.5 (a) Let $S \subseteq \mathbb{R}$. Define an accumulation point of $S$. (1)
(b) Let $S \subseteq \mathbb{R}$. Prove that $S$ is closed if and only if $S$ contains all its accumulation points. (5)

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**Question Four**

4.1 State, without proof, the Nested Interval Property. (2)
4.2 State and prove, the Bolzano-Weierstrass Theorem.

4.3 (a) Let \( \{a_n\} \) be a sequence of real numbers. Define what is meant by “the sequence converges to a real number \( a \)” using \( \epsilon \).
   (b) (i) Prove that if the sequence \( \{a_n\} \) converges to \( a \), then it is bounded.
       (ii) Give a contrapositive of the above statement in (b) (i).
   (c) Show that the seq \( \{a_n : n = 1, 2, \ldots\} \) is unbounded, where \( a_1 = 1; a_2 = 1 + 1/2; a_3 = 1 + 1/2 + 1/3; \ldots\), \( a_n = 1 + 1/2 + 1/3 + \cdots + 1/n \).

4.4 (a) Define a Cauchy sequence.
   (b) Prove that every Cauchy sequence of real numbers is convergent.

Question Five

5.1 Prove that \( \mathbb{Q} = \mathbb{R} \) where \( \mathbb{Q} \) is the set of all rational numbers.

5.2 (a) Let \( f : E \rightarrow R \), where \( E \subseteq R \). Use \( \epsilon \) and \( \delta \) to define what is meant by \( f \) is continuous at \( x_0 \) in \( E \).
   (b) Let \( f \) be continuous at \( x_0 \). Prove that \( f \) is bounded in some neighbourhood of \( x_0 \).
   (c) Let \( f, g : D \rightarrow R \) be continuous functions at \( x_0 \). Prove that (i) \( f + g \) and (ii) \( fg \) are both continuous at \( x_0 \).

5.3 Let \( f : [-5, 3] \rightarrow R \) be given by \( f(x) = \frac{3}{x-2} \). Prove that \( f \) is uniformly continuous on \([-5, 3]\), from definition.

5.4 Let \( D \) be a compact subset of \( R \) and \( f : D \rightarrow R \) be a continuous function. Prove that \( f \) is uniformly continuous.

Question Six

6.1 Show that the equation \( x = \cos x \) has at least one solution in \([0, \frac{\pi}{2}]\).

6.2 (a) State, without proof, the Mean-value Theorem.
   (b) Let \( f : [a, b] \rightarrow R \) be continuous and differentiable on \((a, b)\). Prove
the following statements:
(i) If \( f'(x) \neq 0 \) for all \( x \in (a, b) \), then \( f \) is one-to-one.
(ii) If \( f'(x) = 0 \) for all \( x \in (a, b) \), then \( f \) is constant.
(iii) If \( f'(x) < 0 \) for all \( x \in (a, b) \), then \( x < y \) implies \( f(x) > f(y) \) for \( x \) and \( y \) in \( (a, b) \).

6.3 Give geometric interpretations of statements (i), (ii), (iii) in Question 6.2 (b) above.

6.4 Let \( F = \{\{1, 3\}, \{3, 4\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}\} \). Partially order \( F \) with set inclusion \( \subseteq \). (a) Say, with reasons, whether \( F \) is totally ordered or not; (b) List all the minimal and maximal elements of \( F \), if any.

6.5 (a) Write down the Maclaurin series of \( f(x) = \frac{1}{1-x} \) and determine its region of convergence.
(b) Deduce from (a) above the Maclaurin series for \( \arctan \, x \), stating its region of convergence. Hence use the first 4 terms of the series to approximate \( \arctan \, \frac{2}{3} \) to 4 decimal places.

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