University of Fort Hare

MAT 224

Supplementary Examinations: January 2019

Subject: Mathematics 2
Paper: Real Analysis

Time: 3 Hours Marks: 100 Subminimum: 40

This question paper consists of 4 pages

Internal examiner(s)          External examiner(s)

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Instructions

Answer NO more than FIVE (5) questions. Symbols used have the usual meanings.

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Mathematics Supplementary Exams January 2019

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Question One

1.1 Let $A; B$ and $C$ be sets. Prove that
(a) $A - (B \cap C) = (A - B) \cup (A - C)$;
(b) $A \subseteq B$ if and only if $A \cap B^c = \emptyset$, where $A^c = X - A$. (6)

1.2 If $A, B \subseteq X$, we write $B^c$ for $X - B$ and $A^c$ for $X - A$. Prove that
$A \subseteq B$ iff $B \cup A^c = X$. (3)

1.3 (a) Define what is meant by “set $A$ is equivalent (equipotent) to set $B$”.
(b) Show (from definition) that the interval $(0, 1)$ is equivalent to the set $\mathbb{R}^+$ of all positive real numbers. (1)

1.4 (a) Define what is meant by a “countably infinite set” and hence define
a countable set.
(b) Prove that the set $\mathbb{Q}$ of all rational numbers is countable.
(c) Deduce that the set $\mathbb{Q}^c$ of all irrational numbers is uncountable. (2)

1.5 Let $f : A \rightarrow B$ be an injective function and assume $B$ is countable.
Prove that $A$ is countable. (2)

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Question Two

2.1 (a) Let $S$ be a bounded set. Define what is meant by (i) the supremum
of $S$ and (ii) the infimum of $S$. (3)
(b) Let $S = \{x : 2x^2 - 5x \geq 3\}$. Find (i) $\sup S$ and $\inf S$, if they exist. (3)
2.2 Prove the following: Let $S \subseteq \mathbb{R}$. A real number $a$ is the greatest lower bound of $S$ (i.e. $a = \inf S$) if and only if $a$ is a lower bound of $S$ and for every $\epsilon > 0$ there exists $x \in S$ such that $a + \epsilon > x$.

2.3 Let $x$ be a real number. Prove that there exists an integer $n$ such that $n \leq x < n + 1$.

2.4 Prove the theorem of Eudoxus: Between any two real numbers there is a rational number.

Question Three

3.1 Let $\epsilon > 0$. Prove that $|a| > \epsilon$ if and only if $a > \epsilon$ or $a < -\epsilon$.

3.2 Prove that $||a| - |b|| \leq |a - b|$ for any real numbers $a$ and $b$.

3.3 (a) Define what is meant by “a subset $A$ of $\mathbb{R}$ is open in $\mathbb{R}$”.
   (b) Let $a, b \in \mathbb{R}$ with $b > a$. Prove that the interval $(a, b)$ is an open subset of $\mathbb{R}$.

3.4 (a) Prove that the intersection of a finite number of open sets in $\mathbb{R}$ is open.
   (b) Prove that the union of any number of open sets in $\mathbb{R}$ is open.

3.5 Show from definition that the interval $[a, b]$ is closed in $\mathbb{R}$.

3.6 (a) Define what is meant by “$x$ is an interior point of $A$”.
   (b) Let $A \subseteq \mathbb{R}$. Prove that $A^\circ$ is an open subset of $\mathbb{R}$.

Question Four

4.1 (a) Let $S \subseteq \mathbb{R}$. Define an accumulation point of $S$.
   (b) Let $S \subseteq \mathbb{R}$. Prove that $S$ is closed if and only if $S$ contains all its accumulation points.
4.2 State and prove the Nested Interval Property.

4.3 State, without proof, the Bolzano-Weierstrass Theorem.

4.4 (a) Let \( \{a_n\} \) be a sequence of real numbers. Define what is meant by "the sequence converges to a real number \( a \)" using \( \epsilon \).
   (b) (i) Prove that if the sequence \( \{a_n\} \) converges to \( a \), then it is bounded.
        (ii) Give a contrapositive of the above statement in (b) (i).

4.5 Use \( \epsilon \) and \( \delta \) to show that \( \lim_{x \to 0} \sin(1/x) \) does not exist.

Question Five

5.1 (a) Define a Cauchy sequence.
   (b) Prove that every Cauchy sequence of real numbers is convergent.

5.2 Prove that \( \overline{\mathbb{Q}} = \mathbb{R} \) where \( \mathbb{Q} \) is the set of all irrational numbers.

5.3 (a) Let \( f : E \to R \), where \( E \subseteq R \). Use \( \epsilon \) and \( \delta \) to define what is meant by \( f \) is continuous at \( x_0 \) in \( E \).
   (b) Let \( f \) be continuous at \( x_0 \). Prove that \( f \) is bounded in some neighbourhood of \( x_0 \).
   (c) Let \( f, g : D \to \mathbb{R} \) be continuous functions at \( x_0 \). Prove that (i) \( f + g \) and (ii) \( fg \) are both continuous at \( x_0 \).

Question Six

6.1 Let \( f : [-5, 3] \to R \) be given by \( f(x) = \frac{3}{x^2} \). Prove that \( f \) is uniformly continuous on \([-5, 3]\), from definition.

6.2 Let \( D \) be a compact subset of \( R \) and \( f : D \to R \) be a continuous function. Prove that \( f \) is uniformly continuous.
6.3 (a) State, without proof, the Mean-value Theorem.
(b) Let \( f : [a, b] \rightarrow \mathbb{R} \) be continuous and differentiable on \((a, b)\). Prove the following statements:
(i) If \( f'(x) \neq 0 \) for all \( x \in (a, b) \), then \( f \) is one-to-one.
(ii) If \( f'(x) = 0 \) for all \( x \in (a, b) \), then \( f \) is constant.
(iii) If \( f'(x) < 0 \) for all \( x \in (a, b) \), then \( x < y \) implies \( f(x) > f(y) \) for \( x \) and \( y \) in \((a, b)\).

6.4 Let \( F = \{\{1, 3\}, \{3\}, \{3, 4\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}\} \). Partially order \( F \) with set inclusion \( \subseteq \). (a) Say, with reasons, whether \( F \) is totally ordered or not; (b) List all the minimal and maximal elements of \( F \), if any.

6.5 (a) Write down the Maclaurin series of \( f(x) = \frac{1}{1-x} \) and determine its region of convergence.
(b) Deduce from (a) above the Maclaurin series for \( \arctan x \), stating its region of convergence. Hence use the first 4 terms of the series to approximate \( \arctan \frac{3}{4} \) to 4 decimal places.

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